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A DYNAMIC MODEL OF DEBT

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by

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ABSTRACT

We analyse the role of debt in persuading an entrepreneur to pay out cash flows, rather than to divert them. In the first part of the paper we study the optimal debt contract--specifically, the trade-off between the size of the loan and the repayment--under the assumption that some debt contract is optimal. In the second part we consider a more general class of (non-debt) contracts, and derive sufficient conditions for debt to be optimal among these.

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1. Introduction

Although there is a vast literature on capital structure, economists do not yet have a fully satisfactory theory of debt finance (or of the differences between debt and equity). One of the reasons for this is that debt is a security with several characteristics: a debtor typically promises a creditor a noncontingent payment stream; provides the creditor with the right to foreclose on the debtor's assets in a default state; and gives the creditor priority in bankruptcy. It is unclear whether all these characteristics are equally important, and whether they necessarily have to go together. In this paper we develop a model based on the second characteristic of debt -- the foreclosure right -- although our model implicitly has something to say about the other two characteristics as well.

We consider an entrepreneur who needs funds from an investor (e.g., a bank) to finance an investment project. The project will on average generate returns in the future, but these returns accrue to the entrepreneur in the first instance, and cannot be allocated directly to the investor. We consider the stark -- and extreme -- case where the entrepreneur can "divert" or "steal" the project returns on a one-for-one basis. However, the entrepreneur cannot "steal" the assets underlying the project. Under these conditions we show that a debt contract of the following form has value. The entrepreneur promises to make a fixed stream of payments to the investor. As long as he makes these payments, the entrepreneur continues to run the project. However, if the entrepreneur defaults, the investor has the right to seize and liquidate the project assets. At this stage the entrepreneur and investor can renegotiate the contract.

Our model supposes symmetric information between the entrepreneur and investor both when the contract is written and once the relationship is under way. However, many of the variables of interest, such as project returns and asset liquidation value, are assumed not to be verifiable by outsiders, e.g., a court; hence contracts cannot be conditioned (directly) on these. The symmetry of information between the parties means that renegotiation of the debt contract following default is relatively straightforward to analyze. However, renegotiation does not necessarily lead to first-best efficiency. The reason is that situations can arise where even though the value to the entrepreneur of retaining assets exceeds their liquidation value, there is no credible way for the entrepreneur to compensate the investor for not liquidating the assets. The point is that the entrepreneur may not have sufficient current funds for such compensation (particularly if his loan default was involuntary), and, while he may promise the investor a large fraction of future receipts, the investor will worry that when the time comes, she will not be able to get her hands on these: the entrepreneur will default. Thus inefficient liquidation may occur in equilibrium.

To simplify matters, we restrict attention to the case where the entrepreneur-investor relationship lasts for just two periods (or three dates). That is, we suppose the entrepreneur requires funds at date 0, a return is realized at date 1, and if the project is continued, a further return is earned at date 2. We also assume that part or all of the project can be liquidated at date 1 and that project returns can be reinvested. Let the entrepreneur's wealth be w and the cost of the project be $I > w$. Then a debt contract is characterized by two numbers (P, T), $T \geq 0$, where $I - w + T$ is the amount the entrepreneur borrows at date 0 and P is the promised repayment at date 1. (It is easy to show that the entrepreneur will pay nothing at date 2.)

In Section 3 we explore the trade-off between P and T. The more the entrepreneur borrows at date 0, the more he must repay at date 1, i.e., there is a positive relationship between the two variables. Each instrument has a different role to play, however. The advantage of a low value of P is that it strengthens the entrepreneur's position in good states of the world by giving him the right to continue using the assets in exchange for a small repayment. This prevents the investor from using her bargaining power to liquidate assets when they are worth a lot to the entrepreneur. The advantage of a high value of T is that it strengthens the entrepreneur's position in bad states of the world, i.e., default states, by giving him additional liquidity. This allows the entrepreneur to repurchase assets from the investor in the renegotiation process.

In general, it is optimal to use both instruments. However, in Propositions 1-3 we obtain sufficient conditions for just one instrument to be used. We show that, depending on the stochastic structure of the problem, a "simple debt" contract or a "rental" contract will be optimal. A simple debt contract is one where $T = 0$, that is, the entrepreneur borrows the minimum amount necessary to finance the project (in other words, there is "maximum equity participation"). At the other extreme, a "rental" contract is one where the entrepreneur borrows the maximum amount possible at date 0 and defaults with certainty at date 1, i.e., P is as high as possible (in effect, the entrepreneur rents the assets between dates 0 and 1).

Sections 2 and 3 are based on the assumption that a debt contract is optimal for the entrepreneur and investor. In Section 4 we examine this assumption. Are there other contracts that can solve the cash diversion problem with greater efficiency? In general the answer is yes. One intepretation of a debt contract is that it provides the entrepreneur with the right to continue the project if he makes a prespecified payment at date 1. An alternative contract would give the investor an option (or right) to liquidate the project if she makes a prespecified payment at date 1. More complicated contracts may also be useful. For example, the right to continue the project could be a (stochastic) function of how much the entrepreneur pays. More generally, the entrepreneur and investor could agree to play a message game whereby the amount each party has to pay, and the allocation of the right to control the project assets, are functions of verifiable messages sent by the two parties at date 1.

In Section 4 we show that, under some reasonable assumptions, the additional complexity provided by messages is unnecessary. That is, a debt contract is optimal within a large class of (message-game) contracts. The conditions required for this result are that reinvestment in the project at date 1 yields the same rate of return as the project itself (i.e., the project exhibits constant returns to scale at date 1); and that the project returns at dates 1 and 2 and the liquidation value are positively related. In fact, under these conditions, we show that simple debt is optimal, i.e., within the class of debt contracts it is optimal to set $T = 0$.

There is a simple intuition for the optimality of (simple) debt. Ex post, every dollar that the investor receives is a dollar that the entrepreneur cannot reinvest. Under the assumption that the project exhibits constant returns to scale at date 1, and that the key return and liquidation variables are correlated, it is desirable to maximize the entrepreneur's resources in "good" (high return) states of the world, and -- given that the investor must be repaid -- maximize the investor's payoff in "bad" (low return) states of the world. The reason is that this enables the entrepreneur to reinvest as much as possible when reinvestment is most valuable. Debt does a good job of achieving this since it puts a cap, P, on the investor's payoff by giving the entrepreneur the right to continue using the assets if he pays P. This cap will be binding in good states of the world, thus limiting the investor's payoff and maximizing the entrepreneur's resources. In contrast, a contract that, say, gives the investor the option to liquidate the project has exactly the opposite (and wrong) effect: the investor will buy out the entrepreneur when the project assets are worth a lot, which means that the investor's return is high in good states of the world.

We have visited some of the themes of this paper in previous work. Hart and Moore (1989) provides an early version of the model and also contains a preliminary extension to the case of more than two periods. Unfortunately, the multiperiod case is far from straightforward except when there is perfect certainty. For an analysis of the multiperiod certainty case, and a discussion of its implications for the maturity structure of debt contracts, see Hart and Moore (1994) and Hart (1995); the former contains a slight variant of the model presented here -- the entrepreneur can quit, that is, withdraw his human capital from the project, rather than divert the project returns.

The paper is organized as follows. The model is presented in Section 2. Section 3 analyzes the optimal choice of P and T. Section 4 considers more general contracts. Section 5 allows for the possibility of variable project scale at date 0. Finally, Section 6 discusses the relationship of our work to the literature, and contains some concluding remarks.

2. The Model

We consider a risk neutral entrepreneur who requires finance for an investment project. The project costs I and the entrepreneur's initial wealth is $w < I$. There is a competitive supply of risk neutral investors, each of whom is prepared to finance the project as long as she breaks even. The task for the entrepreneur is to design a payback agreement that persuades one of them to put up at least $(I - w)$ dollars.[1]

The project lasts two periods, as illustrated in Figure 1, with returns being generated at dates 1 and 2. These returns, which will typically be uncertain as of date 0, are specific to this entrepreneur; that is, they cannot be generated without his cooperation. For simplicity, however, we ignore any actions taken by the entrepreneur to generate them; that is, the returns are produced simply by his being in place.

FIGURE 1 NEAR HERE

As emphasized in the Introduction, the project returns R_1, R_2 accrue to the entrepreneur in the first instance. Thus, the payback agreement must be designed to give the entrepreneur an incentive to hand over enough of these returns to the investor to cover her initial cost. We take the entrepreneur's and investor's discount rates both to be zero, which is also the market interest rate.

The I dollars of investment funds are used to purchase assets which at date 1 have a second-hand or liquidation value $L > 0$, whose expectation, EL, is less than I. We suppose the assets are worthless at date 2.

We also assume that any funds not paid over to the investor at date 1 can be reinvested in the project. These funds earn a rate of return equal to s between dates 1 and 2, where $1 \le s \le R_2/L$. That is, at worst reinvestment yields the market rate of interest and at best it yields the same rate of return as the initial project itself. We allow both s and L to be random variables as of date 0 (along with R_1 and R_2). Note that the assumption $1 \le s \le R_2/L$ implies that the project's going-concern value (R_2) is at least as high as its liquidation value (L) at date 1.

We make some further assumptions. First, all uncertainty about R_1, R_2,

[1] We ignore agreements with several investors. But see Section 6.

L and s is resolved at date 1.[2] Second, as a result of their close post-investment relationship, both parties learn the realizations R_1, R_2, L and s at this date (so they have symmetric information). However, these realizations are not verifiable to outsiders, and so date 0 contracts cannot be conditioned on them (at least directly).

Third, the assets are divisible at date 1. That is, if a fraction $(1 - f)$ of the assets is sold off at date 1, then the date 1 liquidation receipts will be $(1 - f)L$ and the date 2 project return will be fR_2.

Finally, we assume that the project is productive, in the sense that it would be carried out in a first-best world. If $s > 1$ with positive probability, this is always the case since the project is a "money pump" at date 1 (one dollar at date 1 yields $s > 1$ dollars at date 2). If $s \equiv 1$, then the required condition is $E[R_1 + R_2] > I$, i.e., the project has positive expected net present value in the absence of reinvestment.

Feasible Contracts

We assume that, as the cash flows R_1 and R_2 accrue to the entrepreneur, he can divert them for his own benefit.[3] In contrast, the physical assets (those purchased with the initial investment funds) are fixed in place and can be seized by the investor in the event of default.[4] In addition, seizure

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[2] This is without loss of generality since we can always replace the realization of a random variable by its expected value.

[3] This (admittedly extreme) assumption is meant to capture the idea that the entrepreneur has discretion over cash flows. One way the entrepreneur might divert cash flows is by selling the output from this project to another firm he owns at an artificially low price or by buying input from another firm at an artificially high price.

[4] In practice the distinction between cash flows (which can be diverted) and physical assets (which cannot) may not be as stark as we assume. What is

is the worst outcome that can befall the entrepreneur. That is, we rule out jail or physical punishment as ways of disciplining a nonperforming entrepreneur.[5]

Given that the entrepreneur can divert the cash flows, but not the project assets, it is natural to consider the following debt contract. The entrepreneur (henceforth known as the debtor D) borrows $B \geq I - w$ at date 0 and agrees to make fixed payments at dates 1 and 2; and if he fails to do so the investor (henceforth known as the creditor C) can seize the project assets.

We will find it convenient to write $B = I - w + T$, where $T \geq 0$ can be interpreted as the "transfer" that D receives from C, over and above what he needs to finance the project. It is assumed that D places this transfer in a (private) savings account. That is, T represents non-recourse financing (it cannot be seized by the creditor).[6]

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important for the analysis that follows is that the investor can get her hands on something of value in a default state: the physical assets represent this source of value. Obviously, if the entrepreneur can divert everything, including the assets which generate future cash flows, then the investor has no leverage at all.

[5]One justification for ruling out jail is that there is always enough background uncertainty so that the entrepreneur can claim that $R_1 = R_2 = 0$ (recall that R_1 and R_2 are not verifiable). Hence it would be difficult to persuade a judge or jury to convict the entrepreneur of theft. A justification for ruling out (private) physical punishment -- apart from the fact that it is probably illegal -- is that the investor has no incentive to administer the punishment ex post if it is at all costly (she gains nothing from it), i.e., punishing the entrepreneur is not credible.

[6]If T is put in a "public" rather than a "private" savings account, i.e., if it can be seized by the investor, then a positive T is equivalent to a lower value of P. Thus this case does not have to be considered.

It is clear that there is no way to persuade D to pay anything at date 2, since at that stage the assets are worthless and so C has no leverage over D. Hence, we can set the date 2 payment equal to zero. From now on, therefore, we write the date 1 payment as P and denote a debt contract by a pair (P,T).

C and D's payoffs conditional on the state (R_1, R_2, L, s)

Suppose that a debt contract is in place and a particular realization (R_1, R_2, L, s) of the return streams and liquidation value occurs at date 1. How will D react? Note that D's wealth at date 1 is $T + R_1$, since he carries over T from date 0 and the project has earned R_1; moreover, all of this is in a private savings account, i.e., it can be diverted. In contrast, the project has assets, with a liquidation value of L, that can potentially be seized by C.

We will assume that D can pay C either from his private savings account or by liquidating project assets. That is, even though D cannot divert or steal project assets for his own purposes, he can use them for debt repayment purposes. We discuss this assumption further in footnote 11 below. Note that, since $s \le R_2/L$ (i.e., the initial project has a higher rate of return than does reinvestment), D will never liquidate assets if he has cash in hand. That is, liquidation is a last resort.

Thus, if $T + R_1 + L \ge P$, D has two choices: either he can make the payment P, or he can default (voluntarily), i.e., pay zero.[7] In contrast, if $T + R_1 + L < P$, D has only one choice: to default (involuntarily).

In the event of default, C has the right to seize the project assets. However, seizure is only a threat point. If the liquidation value L is low,

[7]It is easy to show that it is never in D's interest to make a partial payment.

C may prefer to renegotiate the debt contract.

We will adopt a simple form of renegotiation. In the basic renegotiation game, we suppose that with probability $(1 - \alpha)$ D makes a take-it-or-leave-it offer to C and with probability α C makes a take-it-or-leave-it offer to D.[8] It turns out that, because the set of feasible payoffs is convex, the randomness in this game can cause inefficiency (this will be clear from Figure 2 below). Thus, we modify the basic game in one respect: we allow D to make C an offer before the game starts. The inefficiency is thereby eliminated.

We begin by computing C's payoff under renegotiation. If D makes a take-it-or-leave-it offer to C, then C's payoff is L, which is what she would get if she turned down D's offer and liquidated the assets. The situation where C makes a take-it-or-leave-it offer is more complicated. We divide this into two sub-cases, according to whether $T + R_1 < R_2$ or $T + R_1 > R_2$.

Suppose first that $T + R_1 < R_2$ (D is "poor"). Then C will ask for all of D's cash $T + R_1$, and will also insist that a fraction $1 - \left(\frac{T + R_1}{R_2}\right)$ of the assets be liquidated. In return D will be handed back the remaining fraction $f = \frac{T + R_1}{R_2}$. This makes D's payoff $T + R_1$, which is what he would get if he rejected C's offer. C's return is given by

$$T + R_1 + \left[1 - \left(\frac{T + R_1}{R_2}\right)\right]L.$$

Suppose next that $T + R_1 \geq R_2$ (D is "rich"). Then C will agree to sell back the project assets to D in return for a cash payment of

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[8] In Hart and Moore (1989) a different bargaining process was considered. This turned out to imply $\alpha = 1$.

$$T + R_1 - \left(\frac{T + R_1 - R_2}{s}\right).$$

This leaves D with cash equal to $\left(\frac{T + R_1 - R_2}{s}\right)$, which, when reinvested at the rate of return s, and added to the project return R_2, gives D a total payoff of $T + R_1$. Again this is what D would get if he rejected C's offer.

We can combine these two subcases to write C's payoff, when C has all the bargaining power, as

$$\min\left\{T + R_1 + \left[1 - \left(\frac{T + R_1}{R_2}\right)\right]L,\quad T + R_1 - \left(\frac{T + R_1 - R_2}{s}\right)\right\}.$$ [9]

To obtain C's overall (expected) payoff, $\overline{P}$, in the renegotiation game, we weight C's payoff when D has all the bargaining power and C's payoff when C has all the bargaining power by the probabilities with which they occur. This yields

(2.1) $\overline{P}(R_1,R_2,L,s;\ T)$ =

$$(1 - \alpha)L + \alpha\min\left\{T + R_1 + \left[1 - \left(\frac{T + R_1}{R_2}\right)\right]L,\quad T + R_1 - \left(\frac{T + R_1 - R_2}{s}\right)\right\}.$$

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[9] To understand the min formula, note that the two terms are equal when $T + R_1 = R_2$, and that the coefficient of $T + R_1$ is smaller in the second term than in the first. Hence, the second term is bigger than the first term when $T + R_1$ is low; however, this is when C's payoff is given by the first term.

So far we have assumed that D defaults. However, if D can pay his debt (i.e. if $T + R_1 + L < P$), he may choose to do so, in which case C's payoff is P, as opposed to $\overline{P}$. We will show later (following Figure 2) that D will adopt the strategy that minimizes C's payoff. In other words, D will pay P if and only if $P \leq \overline{P}$. Note that this rule also covers the case where D is forced to default (i.e. where $T + R_1 + L < P$), because $\overline{P} \leq T + R_1 + L$, which implies $\overline{P} < P$.

Hence C's gross payoff is $\min\{\overline{P}, P\}$, and her payoff net of the initial transfer T equals

$$(2.2) \qquad g(R_1,R_2,L,s;\ P,T) \;=\; \min\{\overline{P} - T,\ P - T\},$$

where $\overline{P}$ is given by (2.1).

The final step is to calculate D's payoff. As we have noted, it is always efficient for D to pay C in cash if he can. Thus, if $T + R_1 \geq \min\{\overline{P}, P\}$, D will pay C entirely in cash. Since D retains the project assets and reinvests the remaining cash at the rate of return s, his payoff is

$$R_2 \;+\; \left(T + R_1 - \min\{\overline{P},\ P\}\right)s.$$

On the other hand, if $T + R_1 < \min\{\overline{P}, P\}$, then D will hand over all his cash and liquidate a fraction $(1 - f) = \left(\min\{\overline{P}, P\} - T - R_1\right)/L$ of the assets to realize the remainder of C's payoff. In this case, D's payoff, fR_2, equals

$$\left(1 \;-\; \frac{\min\{\overline{P},\ P\} - T - R_1}{L}\right)R_2.$$

We can combine these two expressions and use (2.2) to obtain the following formula for D's payoff:

$$(2.3) \qquad h(R_1,R_2,L,s;\ P,T) = \min\left\{R_2 + (R_1 - g)s,\ R_2 + (R_1 - g)\frac{R_2}{L}\right\}.^{10}$$

The fraction of the project assets that D retains equals

$$(2.4) \qquad f(R_1,R_2,L,s;\ P,T) = \min\left\{1,\ 1 + \frac{R_1 - g}{L}\right\}.$$

(2.1) - (2.4) summarize the situation at date 1, conditional on the state (R_1,R_2,L,s) and the debt contract (P,T). In particular, (2.3) expresses the relationship between D's and C's payoff. For a given state of the world and a given value of T, Figure 2 graphs this relationship, as P varies between 0 and ∞.

FIGURE 2 NEAR HERE

Figure 2 justifies our earlier assertion that D will pay P if and only if $P \leq \bar{P}$. The frontier is downward-sloping and so lower levels of C's payoff imply higher levels of D's payoff. Hence D always wants to minimize C's payoff. Note that, if $P \leq \bar{P}$, it is always feasible for D to keep C's payoff down to P: he can simply mimic the outcome of the renegotiation game but hand over less cash and/or liquidate fewer assets.[11]

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[10]To understand this min formula, note that the two terms are equal when $T + R_1 = \min\{\bar{P}, P\}$ (i.e., when $g = R_1$), and that the coefficient of g is less negative in the first term than in the second. Hence the first term is bigger than the second term when g is high; however, this is when D's payoff is given by the second term.

[11]The assumption that D can liquidate project assets by himself to pay C is

We close the section with a numerical example. Suppose $T = 3$, $R_1 = 9$, $R_2 = 18$, $L = 6$, $s = 1$ and $\alpha = 1$ (C has all the bargaining power). From (2.1), $\bar{P} = 14$: in a default state C, sells 2/3 of the assets, which are worth 18, back to D for 12 and liquidates 1/3 for an extra 2. Thus if $P > 14$, D defaults; C receives 14 (or 11, net of T); and $f = 2/3$. On the other hand if $P \leq 14$, D pays P; C receives P (or $P - 3$, net of T); and $f = 1$ if $P \leq 12$ (D pays C entirely in cash), or $f = (18 - P)/6$ if $12 < P \leq 14$ (D pays C partly in cash and partly by liquidating assets).

We can use this example to stress the fundamental source of inefficiency in the model. Take any case where $P > 12$, so that liquidation occurs ($f < 1$). Then, relative to first-best, this is inefficient since both parties could be made better off by setting $f = 1$ and compensating C out of the additional revenues at date 2. The problem is that there is no credible way for D to compensate C at date 2: C knows that, whatever promises have been made, D will default at date 2 since the assets are worthless then.

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crucial here. If D could not self-liquidate, then a situation might arise where $T + R_1 < P < \bar{P}$, but C's payoff would be $\bar{P}$ rather than P since D would be forced to default.

There are two justifications for the assumption that D can self-liquidate. The first is that C's loan is secured on the general assets of D's company, rather than on specific assets, and that D can sell these general assets for cash in the normal course of doing business (i.e., it would be prohibitively expensive for C to monitor every transaction in which D is engaged). A second justification is the following. Suppose the loan is secured on specific project assets (and these are registered and cannot be sold). Then D could always rent the assets to a third party between dates 1 and 2. C would not need to be aware of this since the third party could ensure that D used the proceeds to pay C at date 1, i.e., D would not be in default. Moreover, if D defaults at date 2 and the assets (which are now worthless) are handed to C, then this does not affect the third party since he has already had the use of them between dates 1 and 2.

3. Analysis of the Optimal Debt Contract

We turn next to the optimal choice of P and T. Since D and C are risk neutral, an optimal contract will maximize D's expected return subject to the constraint that C's expected gross return is no less than I - w + T (the amount borrowed by D). In other words, an optimal contract solves:

$$(3.1) \qquad \max_{P,T \geq 0} \; Eh \qquad \text{s.t.} \quad Eg \geq I - w,$$

where g and h (indexed by the state and the debt contract) are given by (2.2) and (2.3), and the expectations are taken with respect to the joint distribution of R_1, R_2, L and s. Note that C's break-even constraint will hold with equality at the optimum since otherwise D's expected payoff could be increased by lowering P or raising T.

The following lemma is useful.

Lemma 1. (1) $\bar{P}(R_1,R_2,L,s;\, T) - T$ is decreasing in T. (2) $g(R_1,R_2,L,s;\, P,T)$ is increasing in P and decreasing in T. (3) $g(R_1,R_2,L,s;\, P,T)$ falls when P and T rise by the same amount.

Part (1) follows directly from the formula for $\bar{P}$ in (2.1). It reflects the fact that if C hands over an extra dollar at date 0, she will get only part of this back at date 1 in debt renegotiation. Parts (2) and (3) follow directly from part (1), given that $g = \min\{\bar{P} - T,\, P - T\}$.[12]

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[12]We can use Lemma 1 to understand when the project will be undertaken in a second-best world. A necessary and sufficient condition for the project to be undertaken is that the constraint set in (3.1) is non-empty and the maximized value of the objective function exceeds w (which is what D would obtain if the project did not go ahead). Since g is increasing in P and

We turn now to the optimal mix of P and T. An inspection of (2.2) and (2.3) reveals that the two instruments P and T have distinct roles. On the one hand, a dollar reduction in P increases D's payoff in nondefault states, that is, in states where $P \leq \overline{P}$ (D's payoff increases by s if f = 1 and by R_2/L if f < 1). On the other hand, a dollar increase in T increases D's payoff in all states (again by s if f = 1 and by R_2/L if f < 1). In other words, a reduction in P helps D in nondefault states exclusively; whereas an increase in T helps D in all states, and hence, in relative terms, helps D in default states.

However, as part (3) of Lemma 1 shows, a dollar increase in T reduces C's payoff by more than a dollar decrease in P. So, from D's perspective, the trade-off is between a relatively small increase in T and a relatively large decrease in P.

We now present some propositions showing how each instrument can be useful in different circumstances. We will find it useful to define two "polar" debt contracts.

Definition. A "simple debt" contract is a debt contract where T = 0. A "rental" contract is a debt contract where $P = \infty$.

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decreasing in T, C's net return is maximized when $P = \infty$ and T = 0, that is, it equals $E\overline{P}$. It follows that $E\overline{P} \geq I - w$, i.e.,

$$(*) \quad (1 - \alpha)EL + \alpha E\text{Min}\{R_1 + (1 - R_1/R_2)L,\ R_1 - (R_1 - R_2)/s\} \geq I - w$$

is a necessary condition for the constraint set to be non-empty; hence (*) is a necessary condition for the project to take place. When w = 0, (*) is also sufficient since D's participation constraint is non-binding. It is clear from an inspection of (*) that some profitable projects will not be carried out.

In a simple debt contract, D borrows the minimum amount necessary to finance the project (to put it another way, D puts in all his wealth, so that there is full equity participation). In a rental contract D has the right to use the project assets for only one period; at date 1 control reverts to C (in this sense he rents the assets). To put it another way, in a rental contract, D borrows so much that he defaults with probability 1 (for this, all that is required in the finite state case is that P be high; P does not have to equal ∞).[13]

<u>Proposition 1.</u> Suppose either (1) R_1, R_2, L and s are nonstochastic, or (2) L is nonstochastic and $\alpha = 0$. Then all debt contracts that satisfy C's break-even constraint with equality are equally good. In particular, a simple debt contract is optimal.

Proposition 1 follows from the fact that, if (1) or (2) holds, $\bar{P}$ in (2.1), and hence C's payoff g in (2.2), are constants. Hence, given that C breaks even, $g \equiv I - w$, and so D's payoff h in (2.3) is independent of P and T.

The next proposition relates to the special case $s \equiv 1$. Here, the total social surplus from the project -- i.e., the sum of D and C's ex post

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[13] Another interpretation of a rental contract is that C is the owner of the project at date 1 and so can make the decision about whether to continue or liquidate the project. In this case T can be understood as an upfront payment C makes to D at date 0 in exchange for the ownership rights she acquires.

As this interpretation makes clear, there is a role for outside (voting) equity (or ownership) in the model. However, there is no role for nonvoting equity, since D will never voluntarily pay a dividend to shareholders (he prefers to divert all the cash flows).

payoffs, g + h -- equals $R_1 + fR_2 + (1 - f)L$. Since the solution to (3.1) must maximize Eg + Eh s.t. $Eg \geq I - w$ (given that $Eg = I - w$ at the optimum), it follows that, when $s \equiv 1$, an optimal contract solves:

$$\text{(3.2)} \qquad \max_{P,T \geq 0} \quad E[f(R_2 - L)]$$

$$\text{s.t.} \quad Eg = I - w.$$

In other words, an optimal contract as far as possible concentrates any liquidation onto those states where $R_2 - L$ is low.

Proposition 2. Suppose $s \equiv 1$. Then: (1) If only R_1 is stochastic, a rental contract is optimal. (2) If only R_2 is stochastic, a simple debt contract is optimal. (3) If only L is stochastic and $\alpha = 1$, a rental contract is optimal.

Proof. See Appendix.

The following two examples illustrate, and provide intuition for, parts (1) and (2) of Proposition 2.

Example 1. In this example, only R_1 is stochastic. Assume I = 20, w = 7, R_2 = 18, L = 6, s = 1, R_1 = 21 with probability 1/2 and 9 with probability 1/2. Assume $\alpha = 1$ (i.e., C has all the bargaining power).

Consider first the rental contract which causes C to break even. This is given by (P, T) = (∞, 3). When R_1 = 21, the debt is renegotiated down to $\overline{P}$ = 18, and D pays this amount in cash. When R_1 = 9, the debt is renegotiated down to $\overline{P}$ = 14; D pays his cash holding of T + R_1 = 12, and 1/3 of the assets are sold to make up the difference. C's expected payoff, net of the transfer $T = 3$, is $(15 + 11)/2$, which covers her initial investment of $I - w = 13$. In the low R_1 state there is inefficient liquidation ($f = 2/3$). D's expected return $= (1/2)[24 - 18 + 18] + (1/2)[12 - 12 + (2/3)18] = 18$.

Now at the opposite extreme consider the simple debt contract $(P, T) = (14, 0)$. When $R_1 = 21$, D pays the debt of 14 (his cash holding of 21 is adequate to meet the debt payment; moreover, $\overline{P} = 18 > 14$ and so D doesn't wish to default). When $R_1 = 9$, the debt is renegotiated down to $\overline{P} = 12$; D pays his cash holding of 9, and 1/2 of the assets are sold to make up the difference. C's expected payoff is $(14 + 12)/2$, which covers her initial investment. In the low R_1 state there is inefficient liquidation ($f = 1/2$) -- more so than for the rental contract. D's expected return $= (1/2)[21 - 14 + 18] + (1/2)[9 - 9 + (1/2)18] = 17$.

So the simple debt contract is inferior to the rental contract $(P, T) = (\infty, 3)$. (Indeed, Proposition 2(1) tells us that the rental contract dominates any contract with $T < 3$.)

The intuition behind this example (and behind Proposition 2(1) more generally) is the following. When R_1 is high, D is wealthy and so there will not be much liquidation (f is increasing in R_1 from (2.4)). Thus f is likely to equal 1 in high R_1 states and be less than 1 in low R_1 states. Helping D in high R_1 states therefore does not contribute to social surplus (see (3.2)). Thus it is important to target the low R_1 states. But these are the states where default occurs (since $\overline{P}$ is increasing in R_1). Hence a rental contract, which helps D in the default states through a high T, is more effective than a simple debt contract.

<u>Example 2.</u> In this example, only R_2 is stochastic. Assume $I = 20$, $w = 11$, $R_1 = 12$, $L = 6$, $s = 1$, $R_2 = 24$ with probability 1/2 and 8 with probability 1/2. Assume $\alpha = 1$.

The simple debt contract which causes C to break even is $(P, T) = (10, 0)$. When $R_2 = 24$, D pays the debt of 10 (his cash holding of 12 is adequate to meet the debt payment; moreover $\overline{P} > 10$). When $R_2 = 8$, the debt is renegotiated down to $\overline{P} = 8$, and D pays this amount. C's expected payoff is

(10 + 8)/2, which covers her initial investment of I − w = 9. In both states, D retains control of all the assets (f = 1). Thus the first-best is implemented. D's expected return = (1/2)[12 - 10 + 24] + (1/2)[12 - 8 + 8] = 19.

Now, at the opposite extreme, consider the rental contract (P, T) = (∞, 4). When R_2 = 24, the debt is renegotiated down to $\bar{P}$ = 18. D pays his cash holding of 16, and 1/3 of the assets are sold to make up the difference. When R_2 = 8, the debt is renegotiated down to $\bar{P}$ = 8, and D pays this amount. C's expected payoff, net of the transfer T = 4, is (14 + 4)/2 = 9, which covers her initial investment. However, in the high R_2 state, there is inefficient liquidation. D's expected return = (1/2)[16 − 16 + (2/3)24] + 1/2[16 - 8 + 8] = 16.

So the rental contract is inferior to the simple debt contract. (Indeed, Proposition 2(1) tells us that the simple debt contract dominates any contract with T > 0.)

The intuition behind this example (and behind Proposition 2(2) more generally) is the following. When R_2 is high, C can use her bargaining power in the renegotiation process to force a lot of liquidation since even a small fraction of the assets is worth a great deal to D. This creates inefficiency. The best way to prevent this inefficiency is to allow D to keep C at bay by making a low debt payment P; in other words, to help D not to default. But this is precisely what a simple debt contract achieves. In contrast, a rental contract helps D in the default states, i.e. the low $\bar{P}$ states, through the transfer T. However, the low $\bar{P}$ states are also the low R_2 states ($\bar{P}$ is increasing in R_2 in (2.1)), where liquidation is not socially that costly (since R_2 is close to L -- see (3.2)).

We will not provide an example illustrating Proposition 2(3). However, a similar intuition applies. When only L varies, the default states, i.e., the low $\bar{P}$ states, are the states where L is low. These are also the states where liquidation is very costly (since R_2 - L is high). Therefore, a rental contract, which helps D in default states, is good. In contrast, a simple debt contract, which helps D in nondefault states, is less effective.[14]

We conclude this section with a proposition relating to the special case $s \equiv R_2/L$, i.e. where funds that are reinvested yield the same rate of return between dates 1 and 2 as the project itself. In this case, which will be the focus of much of the rest of the paper, we will be able to show that, under a slight strengthening of our assumptions, a simple debt contract is optimal not only among debt contracts, but also relative to a large class of non-debt contracts.

When $s \equiv R_2/L$, the arguments of the min operator in (2.1) are equal. It follows that C's payoff g, net of T, equals

$$\text{(3.3)} \qquad \min\left\{M - T\left(1 - \alpha(1 - \frac{1}{s})\right), \ P - T\right\},$$

where we define the new, derived variable

$$\text{(3.4)} \qquad M \equiv L + \alpha R_1(1 - \frac{1}{s}).$$

And D's payoff equals $sL + sR_1 - sg$. In effect, program (3.1) reduces to

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[14] Proposition 2(3) requires $\alpha = 1$. When $\alpha < 1$ another effect becomes important. A fall in L may reduce $\bar{P}$ so much that D can buy back the assets even when $T = 0$, i.e., there is no liquidation in low L states. (Consider, for example, the case where $\alpha = 0$ and $\bar{P} = L$.) But then a positive T does not improve efficiency in default states and it is better to target the nondefault states through a reduction in P (for an example along these lines, see Example 3 in Chapter 5 of Hart (1995)).

$$(3.5) \qquad \min_{P,T \geq 0} E[gs]$$

$$\text{s.t.} \quad Eg = I - w.$$

In other words, an optimal contract as far as possible concentrates C's payoff onto those states where s is low.

Proposition 3. Suppose $s \equiv (R_2/L)$ and that a higher value of s increases the distribution of M conditional on s, in the sense of first-order stochastic dominance. Then a simple debt contract is optimal.

Proof. See Appendix.

Proposition 3 assumes not only that $s \equiv R_2/L$, but also that increases in s go together with increases in M (and hence $\bar{P}$). This implies that high s states are the nondefault states. Given that high s states are also "good" states where the project assets and reinvestment yield a high rate of return, a simple debt contract, which helps D in nondefault states, works well. In terms of program (3.5), a simple debt contract maximizes C's payoff g in states where s is low.

4. More general contracts

The analysis in Sections 2 and 3 placed considerable restrictions on the class of admissible contracts. We looked only at debt contracts, where D borrows $B = I - w + T$ from C at date 0, and promises to repay a fixed amount P at date 1. In this section we consider a much broader class of contracts. The following example illustrates the power of alternative contracts.

Example 3. Assume $I = 33$, $w = 30$, $L = 20$ and $s = 1$. Both R_1 and R_2 are stochastic; there are three, equally probable, states:

	state 1	state 2	state 3
R_1	43	0	0
R_2	100	320	20

Assume $\alpha = 1/2$ (i.e., C and D have equal bargaining power).

It is straightforward to show that the best debt contract is the simple debt contract $(P, T) = (3, 0)$. Under this contract, D never defaults, and so C's payoff in each of the three states is 3; i.e. the amount D needs at date 0 to finance the project. In state 1, D can pay the debt $P = 3$ from his cash holding of 43, and so there is no liquidation. But in states 2 and 3, D has to liquidate 15% of the assets in order to pay the debt $P = 3$. Although there is no social loss in state 3 (since $R_2 = L$), there is inefficiency in state 2.

The first-best can be achieved, however, by an option-to-buy contract under which C has an option to buy the project assets from D at date 1 at a price $\Pi = 47$. If C exercises her option, the parties will renegotiate, and C's net payoff will be $\overline{P} - \Pi$, where $\overline{P}$ is given by (2.1) with T replaced by Π (reflecting the fact that in the renegotiation D has a total cash holding of $\Pi + R_1$). If C doesn't exercise her option, then C gets nothing and D keeps control over the assets. In state 1, C's net payoff $\bar{P} - \Pi = 56 - 47 = 9 > 0$, and so she will exercise her option. D can pay $\bar{P} = 56$ from his cash holding of $\Pi + R_1 = 47 + 43 = 90$, and there is no liquidation. In states 2 and 3, $\bar{P}$ is less than Π, and so C will not exercise her option; D keeps control over the assets, and, again, there is no liquidation. C's net payoff in states 1, 2 and 3 equals 9, 0 and 0 respectively -- which is worth 3 to her at date 0, i.e., enough to finance D's project.

The option-to-buy contract works well because C is repaid only in state 1, when D has cash; D keeps control in states 2 and 3 without having to pay anything. By contrast, the simple debt contract forces D to pay out in states 2 and 3, when he cannot afford it.[15]

Note that this is not an artifact of the assumption that $s \equiv 1$ in all three states. One can show that in the example an option-to-own contract strictly dominates the simple debt contract (which is the best debt contract), no matter what the values of s, provided s lies in the permitted range $[1, R_2/L]$.

The conclusion one draws from Example 3 is that debt contracts can be strictly inferior to other kinds of contract.

To make further progress, we need to characterise the set of feasible contracts. The option-to-buy contract can be viewed as a special example of a message-game contract, where C sends one of two possible messages at date 1. "Exercising my option" is one of C's messages, the upshot of which is that she owes Π to D, and if she pays she gets control over the assets. "Not exercising my option" is the other message, which leads to D keeping control,

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[15] At the other extreme, the best rental contract (which has $T = 32$) does no better. It can be shown that, in net terms, D has to pay out 3 in state 2 under this contract too. The problem is that although raising P by enough (and giving D the transfer T) serves to subsidize state 2 from state 1, C loses money in state 3 -- which, in ex ante terms, is wasteful. C does not lose money in state 3 under the option-to-buy contract.

and nothing is owed by either party. It is important that the message is public, in that it can be verified by a court in the event of a dispute.

Notice that the contract is effective because the messages provide an indirect way of conditioning on the state of the world. The contract is designed so as to give C the incentive to send different messages in different states. That is, even though a court can't directly verify which state has occurred, C's behavior -- her choice of message -- reveals information about the state.

Once publicly verifiable messages are admitted, the contractual possibilities become rich. There is no reason to limit the set of messages to just two. Also, it needn't be the case that only C sends messages: C and D have common information, and so in principle either of them is in a position to inform the court (indirectly, at least) about the state.

In fact, almost <u>all</u> contracts can be interpreted as message-game contracts. Consider again a debt contract where D owes the amount P. In effect, D has the choice between sending the message "I will pay P and keep control of the assets", or sending the message "I will pay nothing and lose control". As we have seen in Section 2, his choice depends on the state. Thereby, the allocation of control is determined endogenously at date 1.

A richer set of messages from D could make the allocation of control more sensitive to the state. D could send a numerical message: the meaning of message "σ", say, is that he will pay the amount $P = \sigma$ and that, provided he pays, there is then a probability $\rho = \rho(\sigma)$ that he retains control. The lottery $\rho(\cdot)$, which is publicly held, is specified in the date 0 contract. Clearly, there is no loss of generality in restricting attention to nondecreasing functions $\rho(\cdot)$, since D would have no incentive to pay more for a lower probability of keeping control. The more familiar version of this contract is a nonlinear pricing schedule, where D chooses how much to pay, P, and $\rho(P)$ is the probability that he then keeps control (the contract can be thought of as "smoothed debt").

The most general message-game contract we consider is where <u>both</u> C <u>and</u> D send abstract messages -- σ_C and σ_D, say -- at date 1, on the strength of which there is some amount $P = P(\sigma_C,\sigma_D)$ that D owes C. (P may be negative, in which case C owes -P to D.) If the money is paid, then D keeps control over the assets with probability $\rho = \rho(\sigma_C,\sigma_D)$. The mappings $P(\cdot,\cdot)$ and $\rho(\cdot,\cdot)$ are specified in the date 0 contract.[16]

Crucially, however, we continue to assume that, even after message(s) have been sent, D can refuse to pay and choose instead to default. The worst sanction that can be imposed on him is that he loses control of the assets. That is, whatever moves the parties may have made as part of a contractually specified mechanism (whatever messages may have been sent), once some terminal node (P,ρ) has been reached D in effect always has the choice between paying P (if he can afford to) or defaulting -- i.e. choosing the pair (0,0). And if D does default, he can always then renegotiate with C.[17]

We will see in Proposition 4 below that the fact that D can default and renegotiate a contract dramatically reduces the set of message-game contracts that one needs to consider. For the rest of the paper, we look only at the special case $s \equiv R_2/L$.

For this case, we saw in (3.3) that C's net payoff under a debt contract is a function of M and s only, where M is defined in (3.4). In

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[16]There are yet other possible mechanisms, played in stages, which screen on D's cash holdings by requiring him to put up money before he plays a particular branch of the game tree. Such mechanisms exploit infeasibility off the equilibrium path. We postpone discussing these until the end of this section.

[17]In this respect, we depart from much of the literature on implementation, where it is tacitly assumed that agents can be forced to abide by the outcome of a mechanism. It is also usually supposed that the agents can agree in advance not to renegotiate once the mechanism has been played, even though there is no asymmetry of information between them (a usual source of breakdown in bargaining). For an introduction to the literature on implementation in environments with complete information, see Moore (1992).

Lemma 2, we prove that under any message-game contract, C's equilibrium payoff across different states of nature can be expressed in terms of M, s and V, where V is defined by

(4.1) $$V \equiv L + R_1.$$

Accordingly, we can write C's net equilibrium payoff as g(M,s,V). (The third variable, V, separately enters C's payoff only if $\alpha \neq 0$ and, for at least one pair of messages (σ_C,σ_D), the contract specifies $0 < \rho(\sigma_C,\sigma_D) < 1$.)

Notice that M is the most that C can get in the event of D defaulting (and the upper bound is attained only when T = 0). Since D can always default, a corollary is that no message-game contract can give C a net equilibrium payoff greater than M. We formally prove this in (4.2) of Lemma 2.

We actually prove more than this. C's payoff is nondecreasing in each of the three variables M, s and V: see (4.3) in Lemma 2.

Lemma 2. Assume $s \equiv R_2/L$. In any message-game contract, C's equilibrium payoff, net of any transfer T, can be expressed as a function of the three derived variables M, s and V (where M and V are given in (3.4) and (4.1)). Moreover, C's payoff, g(M,s,V) say, must satisfy

(4.2) $g(M,s,V) \leq M$;

(4.3) g(M,s,V) is nondecreasing in M, s and V;

(4.4) g(M,s,V) is independent of V if $\alpha = 0$.

Proof. See Appendix.

As Lemma 2 is a key result in what follows, we should sketch the intuition behind it. In any given state, the two parties are playing a "message/default" game -- after which they will, if necessary, renegotiate their way on to the payoff frontier. The compound game (that is, including the subsequent renegotiation) is akin to a zero-sum game, since the parties' payoffs are perfectly negatively correlated. (It is not a zero-sum game per se, because, unlike in a zero-sum game, the payoff frontier has slope $-s$, not -1. The reason for this is that as C's payoff rises, D has less cash to reinvest and total surplus falls.) In any given state, one can think of this compound game in terms of a reduced-form matrix, where the messages σ_C and σ_D respectively identify the row and column, and the corresponding entry in the matrix specifies a pair of payoffs lying on the frontier. Clearly, C's equilibrium payoff in this compound game cannot be greater than her maximum payoff, M, in any entry of the matrix: hence the upper bound constraint (4.2). (4.3) and (4.4) relate to how C's equilibrium payoff varies with the state. Here we appeal to the fact that the value of the compound game is given by the min-max formula for zero-sum games. Now C's payoff in each entry of the matrix can be shown to be nondecreasing in M, s and V: the point is that an increase in any one of M, s or V increases the surplus, and, for a given (P,ρ), both parties share in the increase. It follows immediately from the min-max formula that C's equilibrium payoff in the compound game is also nondecreasing in M, s and V: hence the monotonicity condition (4.3). Likewise, since C's payoff in each entry of the matrix can be shown to be independent of V if $\alpha = 0$, the same is true of her equilibrium payoff in the compound game: hence the independence condition (4.4).

To sum up what we have learned so far in this section: message-game contracts can be both realistic (e.g. options to buy, or nonlinear pricing contracts) and effective. However, D's ability to default and renegotiate places considerable restrictions on C's equilibrium net payoff. In particular, if $s \equiv R_2/L$, then C's payoff is a nondecreasing function of the three derived variables M, s and V; is bounded above by M; and is independent of V if $\alpha = 0$.

Since D's payoff equals $sV - sg(M,s,V)$, and since $E[sV]$ is independent of $g(\cdot,\cdot,\cdot)$, an optimal message game contract solves the following program, which is akin to (3.5):

$$(4.5)\quad \underset{g(.,.,.)}{\text{Minimize}} \quad E[sg(M,s,V)]$$

$$\text{subject to} \quad E[g(M,s,V)] \;\geq\; I - w;$$

$$g(M,s,V) \;\leq\; M;$$

$$g(M,s,V) \text{ is nondecreasing in } M,\ s, \text{ and } V;$$

$$\text{and} \quad g(M,s,V) \text{ is independent of } V \text{ if } \alpha = 0.$$

The first constraint in (4.5) is C's participation constraint. And by Lemma 2, the last three constraints, (4.2)-(4.4), are necessary conditions on C's equilibrium payoff arising from a message-game contract.[18]

Note that if s is deterministic then it can be seen directly from (4.5) that <u>any</u> contract satisfying the expectation constraint with equality is optimal. (This extends Proposition 1, in the case $R_2/L \equiv s$.)

It is revealing to graph the g-functions that derive from the two contracts which we considered earlier: a debt contract, and an option-to-buy contract. First, consider a debt contract (P,T). C's payoff g is given by (3.3). In Figure 3(a) this g is graphed against M, holding s constant. As we saw in section 3, in order to satisfy C's participation constraint, if T rises by a dollar, then P has to rise by more than a dollar. Accordingly, as T rises, the flat portion of the graph rises, but the vertical intercept

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[18] As we are about to prove that, in certain circumstances, a simple debt contract yields a g function which solves (4.5), the question of sufficiency will not detain us here. In fact, however, one can use the methods of Maskin and Moore (1987) to show that conditions (4.2)-(4.4) are <u>in general</u> sufficient for the existence of a message game in which C's equilibrium payoff is g(M,s,V).

falls. Roughly speaking, a rise in T makes the graph less flat.

FIGURES 3(a) and 3(b) NEAR HERE

Second, consider an option-to-buy contract where C can buy the assets at date 1 by paying Π to D. From (2.1), with $s \equiv R_2/L$, we know that if C exercises her option she can sell the assets back to D for $\bar{P} = M + \alpha\Pi(1 - \frac{1}{s})$. Therefore C will exercise the option if and only if $M + \alpha\Pi(1 - \frac{1}{s}) > \Pi$, and her payoff is given by

$$(4.6) \qquad g(M,s,V) = \max\left\{M - \Pi\left(1 - \alpha(1 - \frac{1}{s})\right),\ 0\right\}.$$

See Figure 3(b), where the function g in (4.6) is graphed against M, holding s constant. Notice that this graph is less flat than the graph for a simple debt contract in Figure 3(a).

Consider the minimand in (4.5). Given C's participation constraint, it would best if her payoff, g, were to decrease in s -- because this would help D in those states when s is high. Moreover, if s is affiliated with M, then we would like g to decrease in M too.[19] Unfortunately, we have to contend with the monotonicity conditions in (4.5). In particular, g has to be nondecreasing in M. The best we can hope for is a flat g, equal to I - w; this corresponds to the riskless simple debt contract P = I - w. However, if there are values of M which are less than I - w, we must respect the upper bound constraint in (4.5). Figures 3(a) and 3(b) suggest that the flattest g corresponds to a simple debt contract. This is confirmed in Proposition 4.

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[19]For a definition and discussion of affiliation, see the Appendix of Milgrom and Weber (1982).

Proposition 4. Assume $s \equiv R_2/L$. If M, s and αV are affiliated then (4.5) is solved by $g(M,s,V) = \min\{M, P\}$, where P satisfies $E[\min\{M, P\}] = I - w$. That is, a simple debt contract, with $T = 0$, is optimal in the class of message-game contracts.

Proof. See Appendix.

A rough intuition for this result was given in the Introduction. Every dollar that C receives at date 1 is a dollar that D cannot reinvest. Under the assumption that $s \equiv R_2/L$, and that the key variables are affiliated, it is desirable to minimize C's payoff, g, in "good" states of the world, since this enables D to reinvest as much as possible when reinvestment is valuable. A simple debt contract works well, since it puts a cap, P, on g, which binds in good states.

The special case $\alpha = 0$ is of independent interest:

Corollary. Assume $s \equiv R_2/L$ and $\alpha = 0$. Then if L and s are affiliated, a simple debt contract is optimal in the class of message-game contracts.

Given $\alpha = 0$, the Corollary tells us that when L and $s \equiv R_2/L$ are affiliated, simple debt is optimal irrespective of R_1. Note that affiliation is implied if either variable is deterministic. Thus, if L is deterministic, then simple debt is optimal irrespective of R_1 or R_2.

This section might be summarised by saying that our exploration of more general contracts -- message-game contracts -- turns out to have been a digression in the case $s \equiv R_2/L$, at least when M, s and αV are affiliated. For this case, we can restrict attention to simple debt contracts after all.

There is a caveat. Although message-game contracts are quite general, there are other forms of mechanism which are played in stages and which screen on D's date 1 cash holdings by requiring him to put up money early on, before he plays a particular branch of the game tree. In effect, these mechanisms exploit infeasibility off the equilibrium path.

For example, suppose that $I = 1 > w$, and that at date 1 there are two equally likely states 1 and 2: $(L, R_1, R_2, s) = (2,0,2,4)$ in state 1; and $(L, R_1, R_2, s) = (2,1,10,20)$ in state 2. Also assume that $\alpha = 0$. (Note that M, s and αV are affiliated, so Proposition 4 applies.) Consider (4.5), but without the constraint that g is nondecreasing in M and s. It is easy to see that the solution is $g = 2$ in state 1 and $g = -2w$ in state 2. This solution violates the constraint that g is nondecreasing in M and s, and hence cannot be achieved by any of the message games described in this section.

However, the following mechanism <u>does</u> achieve the above solution. Let the contract specify that if D pays 3 to C at date 1, then C must pay $3 + 2w$ to D. However, if D fails to pay 3, then C obtains control of the assets. This contract achieves the desired outcome because in state 2 D can pay 3, whereas in state 1 he cannot.

There is an obvious problem with a mechanism like this. In state 1, D could approach a third party and borrow (short-term) using as collateral the payment he is about to receive from C. We suspect that, if this kind of borrowing is allowed, all that matters is the <u>net</u> amount, P, that D is required to pay -- which brings us back to message games with final outcomes (P, ρ), which we have considered. However, these matters require further investigation.

5. Initial Project Scale

We complete our analysis by considering briefly the choice of project scale at date 0. Until now, we have taken the size of the investment, I, to be fixed. Suppose instead that the initial investment can be varied; in particular, suppose that the project exhibits constant returns to scale. It is easiest to think in terms of the "unit project", costing 1 at date 0, with cash returns r_1 and r_2 at dates 1 and 2, and with liquidation value ℓ at date 1. That is, $R_1 = Ir_1$, $R_2 = Ir_2$ and $L = I\ell$. Our concern in this section is with the choice of I.

Given constant returns at date 0, a natural case to consider is where there are also constant returns to scale at date 1: $s \equiv r_2/\ell$. In addition, assume:

(5.1) m, s and αv are affiliated,

$$\text{where} \quad m \equiv \ell + \alpha r_1(1 - \ell/r_2)$$

$$s \equiv r_2/\ell$$

$$\text{and} \quad v \equiv \ell + r_1.$$

We know from Proposition 4 that, given assumption (5.1), simple debt is optimal for any I. Program (3.1), with T = 0, reduces to

$$\text{(5.2)} \qquad \max_{I,P} \; E[Ivs - \min\{Im,P\}s]$$

$$\text{s.t.} \quad E[\min\{Im,P\}] = I - w$$

$$\text{and} \quad I \geq 0.$$

For any I that can be financed, there is some debt level at which C just breaks even. Define P(I) to be the smallest:

(5.3) $\quad P(I)$ is the minimum P satisfying $E[\min\{Im,P\}] = I - w$.

The economics of the problem are revealed by separating D's objective into benefits and costs. Rewrite (5.2) as

(5.4)
$$\max_{I\geq 0} \; Ib - c(I),$$

where $b \equiv E[vs]$

$c(I) \equiv E[\min\{Im,P(I)\}s]$

and $P(I)$ is defined in (5.3).

Notice that D's benefits are linear in I. We show in Proposition 5 below that, among other things, his cost function c(I) is convex.

The case of perfect certainty is particularly simple, and illuminates the stochastic case. From (5.3), $P = I - w \leq Im$. I then solves:

(5.5)
$$\max_{I\geq 0} \; (Iv + w - I)s$$

$$\text{s.t.} \quad Im \geq I - w.$$

There are two cases to consider, $m < 1$ and $m \geq 1$:

(1) $m < 1$. In this case, the constraint in (5.5) eventually binds.

(1a) If $v > 1$, the objective function is increasing in I and there is an interior optimum, $I = w/(1 - m)$. Notice the multiplier if $m > 0$: the optimal scale of the project is proportional to D's initial wealth with a constant of proportionality that exceeds one.

(1b) If $v \le 1$, the objective function is decreasing in I and it is optimal to set $I = 0$.

(2) $m \ge 1$. In this case, the constraint in (5.5) never binds and, since $v \ge m$, the objective function is (weakly) increasing in I. In effect, every increase in I of one dollar increases C's potential payoff by at least 1 dollar, so the project is a money pump and it is optimal to set $I = \infty$.

We now see how these findings generalize when there is uncertainty.

Proposition 5. Assume (5.1) holds, and that there are a finite number of states.

(1) If $Em < 1$ the cost function $c(I)$ defined in (5.4) is increasing, piece-wise linear and convex in the interval $w \le I \le w/(1 - Em)$, with a slope no less than Es.

(1a) If $E[vs] > Es$ then some $I \ge w/(1 - \underline{m})$ is optimal, where $\underline{m}$ is the minimum of m.

(1b) If $E[vs] \le Es$, it is optimal to set $I = 0$.

(2) If $Em \ge 1$ then it is optimal to set $I = \infty$.

Proof: See Appendix.

Proposition 5 tells us that, under assumption (5.1), the problem of choosing I is well-behaved. As with the deterministic case, there is a multiplier, provided $\underline{m} > 0$: if D invests his initial wealth w in the project then he borrows at least a multiple of w for additional investment.

If we were willing to assume a simple debt contract (rather than prove that it is optimal within the class of message-game contracts), assumption (5.1) could be dropped. All the results of the proposition would hold if $E[s|m]$ were nondecreasing in m. The proof in the Appendix makes use of this weaker assumption.

6. Summary, Relationship to the Literature, and Concluding Remarks

A brief summary of the paper may be useful. We have analysed the role of debt in persuading an entrepreneur to pay out cash flows, rather than to divert them. In the first part of the paper we studied the optimal debt contract -- specifically, the trade-off between the size of the loan and the repayment -- under the assumption that some debt contract was optimal. In the second part we considered a more general class of (non-debt) contracts and derived sufficient conditions for debt to be optimal among these.

Our paper can be seen as part of the recent literature that analyzes financial decisions from an "incomplete contracting" perspective. This literature starts with Aghion and Bolton (1992). Aghion and Bolton analyze debt in terms of the allocation of residual control rights over assets (along the lines of Grossman and Hart (1986) and Hart and Moore (1990)). They consider a situation where a project yields private benefits to an entrepreneur as well as (verifiable) monetary benefits. It is assumed that some project actions must be taken in the future, but these cannot be contracted on initially. (One such action might concern the liquidation decision.) If the entrepreneur has all the residual control rights he will take actions that increase his private benefits, but at the expense of the return to investors. On the other hand, if the investor has control she will take actions that do not respect the investor's private benefits. Aghion and Bolton study the optimal balance of control between the entrepreneur and the investor. Of particular interest, they show that the optimal allocation is state contingent: the entrepreneur should have residual control rights in states of the world where his private benefits are relatively high, and the investor should have control in states where the entrepreneur's private benefits are relatively low.

There are two important differences between Aghion and Bolton's work and ours. First, although Aghion and Bolton show that control will shift from the debtor to the creditor in certain states of the world, they do not provide general conditions under which these states can naturally be interpreted as "default" or "bankruptcy" states (for example, they could be high-profit rather than low-profit states). Second, and related, Aghion and Bolton ignore the role of debt as a mechanism for getting a debtor to pay up. That is, Aghion and Bolton assume that control shifts are triggered by a verifiable state of the world (e.g., the state might be that profits are low). In contrast, in our model the shift in control is endogenous -- it occurs because the debtor fails to make a promised repayment.

Our paper also has similarities to Bolton-Scharfstein's (1990) analysis of predation and the costly state verification (CSV) models of Townsend (1979) and Gale and Hellwig (1985). Bolton and Scharfstein develop a model where the penalty for nonpayment of debt is that the creditor withholds future finance rather than liquidating existing assets. They are more concerned with how debt can be used strategically to influence competition in product markets than with a general characterization of debt contracts. In the costly state verification models there is also a penalty for nonpayment, but it is that the debtor is inspected. The CSV models additionally assume that information is asymmetric, tend to rule out ex post renegotiation, and take the cost of bankruptcy as given (it is the cost of monitoring).[20] In contrast, our model is based on symmetric information, allows for ex post

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[20]Gale and Hellwig (1989) does include a discussion of renegotiation, however.

renegotiation and endogenizes the cost of default. For a further discussion of the differences between the CSV models and the incomplete contracting approach, see the Appendix to Chapter 5 of Hart (1995).[21]

There is also a parallel between this paper and the work of Bulow and Rogoff (1989) on sovereign debt. Bulow and Rogoff analyze a model in which a debtor country borrows from a creditor country for current consumption but cannot commit to repay the loan out of future production. If the debtor repudiates the loan, the creditor can retaliate by blockading the debtor country's trade. In the Bulow-Rogoff paper, there is nothing corresponding to irreversible liquidation, and, as a result, there is never any ex post inefficiency (no blockade occurs in equilibrium). In contrast, in our model, there can be inefficient liquidation ex post. Also, because of their concern with sovereign debt, Bulow and Rogoff do not study the role of legally enforceable contracts in sustaining repayment paths.

We conclude by noting some directions for future research. Probably the most interesting extension of the model is to the case of more than two periods, which would permit an analysis of the maturity of debt contracts. As noted in the Introduction, Hart and Moore (1994) and Hart (1995) carry out such an extension, but only for the case of perfect certainty. A preliminary discussion of the uncertainty case was contained in our earlier paper, Hart and Moore (1989). However, the analysis in that paper was intricate; we were unable to go beyond a three-stage model, and there were relatively few clear-cut results. There were some general findings, however, which we believe would broadly apply to any intertemporal model of debt based on

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[21]The work of Allen (1983) and Kahn and Huberman (1988) should also be mentioned. Allen studies a model in which the penalty for not repaying a loan is the seizure of assets and future exclusion from the capital market. However, Allen focuses on inefficiencies with respect to the initial size of the project, rather than on control issues or the cost of default. Kahn and Huberman (1988) investigate the role of asset seizure in encouraging a debtor to repay a loan, but in a context where renegotiation always leads to ex post efficiency.

control. We found that a key tension between short-term and long-term debt is the following. On the one hand, short-term debt gives the creditor early leverage over the project's return stream, which is good because it can keep total indebtedness low. On the other hand, short-term debt may give too much control to the creditor in certain states and lead to premature liquidation; that is, the creditor may liquidate early because the debtor cannot credibly promise to repay later. In this sense, long-term debt contracts protect the debtor from the creditor. An important next step in the research is to formulate a tractable, multiperiod model of debt with uncertainty.

Even in the two-period model there are a number of further avenues to explore. In the first part of the paper we focussed on the trade-off between P and T, and showed that T could be used to limit C's bargaining power in bad states. However, as Section 4 makes clear, more general instruments may be useful when the conditions of Proposition 4 do not hold. One possibility is to give C the right to liquidate only a fraction of the project assets (that is, only some of the project assets would serve as collateral for the loan and the loan would be non-recourse). Another possibility is to give C the right to liquidate the whole project with some probability (this may be particularly useful if the project is indivisible).[22] The whole issue of the role of "non-standard" contracts when the conditions of Proposition 4 do not hold is a challenging topic for future research.

It would also be interesting to relax some of the assumptions we have made about renegotiation. We have supposed that the parties can choose from a large class of mechanisms for allocating control, but that the parties cannot control the division of bargaining power in the renegotiation game. We have also ruled out the presence of third parties to the contract. All these assumptions are worth dropping. For an analysis of how the renegotiation process might be designed to achieve a better outcome, see Harris and Raviv (1995).

In addition, we have studied a one-shot situation. An interesting generalization is to a repeated relationship where parties may acquire a

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[22] The latter idea is explored in Bolton and Scharfstein (1996).

reputation for repaying their debts, or for liquidating assets rather than renegotiating. (A long-lived bank might acquire a reputation for renegotiating only when a default is involuntary.) It would be interesting to know whether under these conditions debt still has a role to play, or whether other instruments might substitute for debt. For an analysis of this and related issues, see Fluck (1996) and Gomes (1996), and, for a more general discussion of debt and reputation, see Diamond (1989).

A further extension is to the case of multiple investors. If there are multiple creditors, then it is plausible that the process of renegotiating a debt contract becomes more difficult (e.g., because the creditors have different information). This brings benefits as well as costs. The benefit is that strategic default by the debtor is less attractive, which means that it is easier to pay the creditor back. The cost is that, if default is involuntary, the project may be liquidated when it should be continued. The trade-off between the two effects is studied in Bolton and Scharfstein (1996).

Finally, in a richer model where the entrepreneur cannot "steal" all the cash flows, (non-voting) equity becomes a feasible claim as well as debt, since dividends can be paid. Dewatripont and Tirole (1994) have shown that under these conditions the entrepreneur's budget constraint can be "hardened" by allocating debt to one outside investor and equity to another. (In a similar vein, Berglof and von Thadden (1994) have shown that it is sometimes optimal to allocate short-term debt to one investor and long-term debt to another.) Incorporating equity into a model like the one described here would greatly enrich the analysis and is an important topic for future research.

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Appendix

Proof of Proposition 2

Consider the set of feasible transfers T. (T is feasible if there exists a debt contract (P,T) satisfying C's participation constraint. We may assume that $T = 0$ is feasible, otherwise the project could not be financed at date 0.) For each feasible T, let P(T) denote the smallest debt level at which C breaks even: $E[g(R_1,R_2,L,s;P(T),T)] = I - w$.

Given $s \equiv 1$, (2.1) reduces to

$$(2.1') \quad \overline{P}(R_1,R_2,L;T) = (1-\alpha)L + \alpha\min\left\{T + R_1 + \left[1 - \left(\frac{T+R_1}{R_2}\right)\right]L,\ \ R_2\right\}.$$

Note that $\overline{P}(R_1,R_2,L;T) - T$ is nonincreasing in T. And by Lemma 1(3), $P(T) - T$ is nondecreasing in T.

First we prove part (1) of the proposition, where only R_1 is stochastic. $\overline{P} = \overline{P}(R_1;T)$ is nondecreasing in R_1. So find $R_1^*(T)$ (which may be infinite) such that $\overline{P}(R_1;T) \le P(T)$ for $R_1 \le R_1^*(T)$ and $\overline{P}(R_1;T) \ge P(T)$ for $R_1 \ge R_1^*(T)$. $R_1 - \overline{P}(R_1;T)$ is also nondecreasing in R_1; so from (2.2), the function $R_1 - g(R_1;P(T),T)$ is nondecreasing in R_1. And $R_1 - g(R_1;P(T),T)$ is nondecreasing in T for $R_1 \le R_1^*(T)$, and is nonincreasing in T for $R_1 \ge R_1^*(T)$.

By (2.4), $f(R_1;P(T),T)$ is a positive affine transformation of $R_1 - g(R_1;P(T),T)$, truncated above by 1 for high R_1. Without the truncation, Ef would be independent of T, since $E[R_1 - g] = ER_1 - I + w$ is independent of T. Thus, with the truncation, Ef rises as T rises. Since $E[f(R_2 - L)] = (R_2 - L)Ef$, it follows from (3.2) that it is optimal to increase T. A rental contract is optimal. Part (1) is proved.

Next we prove part (2) of the proposition, where only R_2 is stochastic. $\overline{P} = \overline{P}(R_2;T)$ is nondecreasing in R_2. So find $R_2^*(T)$ (which may be infinite) such that $\overline{P}(R_2;T) \le P(T)$ for $R_2 \le R_2^*(T)$ and $\overline{P}(R_2;T) \ge P(T)$ for $R_2 \ge R_2^*(T)$.

From (2.2), $g(R_2;P(T),T)$ is also nondecreasing in R_2. And $g(R_2;P(T),T)$ is nonincreasing in T for $R_2 \le R_2^*(T)$, and is nondecreasing in T for $R_2 \ge R_2^*(T)$.

By (2.4), $f(R_2;P(T),T)$ is a negative affine transformation of $g(R_2;P(T),T)$, truncated above by 1 for low R_2. Without the truncation, Ef would be independent of T, since Eg = I - w is independent of T. With the truncation, Ef falls as T rises. Moreover, f falls as T rises when $R_2 - L$ is high. And f rises as T rises when $R_2 - L$ is low. It therefore follows from a standard stochastic dominance argument that $E[f(R_2 - L)]$ falls as T rises. From (3.2), it is therefore optimal to reduce T. A simple debt contract is optimal. Part (2) is proved.

Finally we prove part (3) of the proposition, where only L is stochastic. $\overline{P} = \overline{P}(L;T)$ is nondecreasing in L (when $T + R_1 > R_2$, the second term of the min operator in (2.1′) is strictly less than the first.) So find $L^*(T)$ (which may be infinite) such that $\overline{P}(L;T) \le P(T)$ for $L \le L^*(T)$ and $\overline{P}(L;T) \ge P(T)$ for $L \ge L^*(T)$. From (2.2), $g(L;P(T),T)$ is also nondecreasing in L. And $g(L;P(T),T)$ is nonincreasing in T for $L \le L^*(T)$, and is nondecreasing in T for $L \ge L^*(T)$.

Given $\alpha = 1$, the only way that $g(L;P(T),T)$ can vary with L is because, for at least some $L \le L^*(T)$, the first term of the min operator in (2.1′) is strictly less than the second. In which case it follows that $g(L;P(T),T) > R_1$ for all L. And so, from (2.4), $f(L;P(T),T) < 1$ for all L, and fL - L is a negative affine transformation of g. E[fL - L] is independent of T, since Eg = I - w is independent of T. That is, E[fL] is independent of T. Moreover, fL rises as T rises when L is low, i.e., when $(R_2 - L)/L$ is high. And fL falls as T rises when L is high, i.e., when $(R_2 - L)/L$ is low. It therefore follows from a standard stochastic dominance argument that $E[f(R_2 - L)]$ rises as T rises. From (3.2), it is therefore optimal to increase T.

The other possibility is that $g(L;P(T),T)$ is independent of L. In which case $g(L;P(T),T) \equiv I - w$, which is independent of T.

In sum, a rental contract is optimal. Part (3) is proved.

Q.E.D.

Proof of Proposition 3

Take some debt contract (P,T) for which $T > 0$, and denote C's payoff from (3.3) by $g(M,s)$. Consider replacing this contract by the simple debt contract $(\hat{P},0)$, where $\hat{P}$ is the smallest solution to $E[\min\{M,\hat{P}\}] = I - w$. C's payoff under the latter contract equals $\min\{\hat{P},M\} \equiv \hat{g}(M)$, say, which is independent of s. Given that (P,T) finances I,

$$E[g(M,s)] \geq I - w = E[\hat{g}(M)].$$

This implies that

$$E[\Delta(M,s)] \leq 0, \qquad \text{(i)}$$

where $\Delta(M,s) \equiv \hat{g}(M) - g(M,s)$. It follows that $\hat{P} \leq P - T$.

Now

$$\Delta(M,s) = \begin{cases} T(1 - \alpha + \frac{\alpha}{s}) & \text{for } M \leq \hat{P} \\ \hat{P} - M + T(1 - \alpha + \frac{\alpha}{s}) & \text{for } \hat{P} < M < P - T\alpha(1 - \frac{1}{s}) \\ \hat{P} - P + T & \text{for } M \geq P - T\alpha(1 - \frac{1}{s}). \end{cases}$$

By inspection, $\Delta(M,s)$ is nonincreasing in M and s. Hence taking any $s^- \leq s^+$,

$$\begin{aligned} E[\Delta(M,s^+)|s^+] &\leq E[\Delta(M,s^+)|s^-] \\ &\leq E[\Delta(M,s^-)|s^-]. \end{aligned}$$

Here, the first inequality follows from a standard dominance argument: $\Delta(M,s^+)$ is nonincreasing in M and, for all M, the distribution function of M conditional on s^- is no less than the distribution function of M conditional on s^+. The second inequality reflects the fact that $\Delta(M,s)$ is nonincreasing in s.

Thus we have shown that $E[\Delta(M,s)|s]$ is nonincreasing in s, and so there exists some s*, say, where $1 \le s^* \le \infty$, for which $E[\Delta(M,s)|s]$ is nonnegative for all $s \le s^*$ and is strictly negative for all $s > s^*$. This implies

$$(s - s^*)E[\Delta(M,s)|s] \le 0 \quad \text{for all } s.$$

Taking expectations over s and appealing to the law of iterated expectations,

$$E[s\Delta(M,s)] \le s^*E[\Delta(M,s)],$$

which is nonpositive by (i). Thus $E[s\hat{g}(M)] \le E[sg(M,s)]$. That is, from (3.5), the simple debt contract $(\hat{P},0)$ (weakly) dominates the debt contract (P,T).

Q.E.D.

<u>Proof of Lemma 2</u>

We first need to calculate C's net payoff, γ say, in some state (R_1,L,s) if <u>after</u> having played some mechanism, the parties reach a particular (P,ρ) node with $0 < \rho < 1$. In what follows we suppose that D carries over an amount of cash $T \ge 0$ from date 0.

There are three regions to consider, depending on the size of P: $L + R_1 + T < P$ (region 1); $L < P \le L + R_1 + T$ (region 2); and $P \le L$ (region 3).

In region 1, D does not have enough cash to pay P, and so must default. From (2.1), C's payoff, net of T, is

$$M - T\left(1 - \alpha(1 - \frac{1}{s})\right) = \gamma^1, \text{ say.}$$

In region 2, D can only pay P by augmenting L from his private cash holdings $R_1 + T$. (It is clear that, given $s \equiv R_2/L$ and $\rho < 1$, D will always use the firm's assets, L, in preference to his own.) If D pays P, then with probability ρ he keeps control over the assets, which is an efficient outcome. With probability $1-\rho$, C gets control, in which case they may renegotiate. The renegotiation starts from the status quo: (a) the liquidation value of the remaining assets is zero (since D used them all to contribute L towards the payment P); and (b) D's private cash holdings have gone down to $R_1 + T - (P - L) = V + T - P$. From (2.1), we deduce that, provided D pays P, C's net payoff is

$$P + (1-\rho)\left(\alpha(V + T - P)(1 - \frac{1}{s})\right) - T = \gamma^2, \text{ say.}$$[23]

If $\gamma^2 > \gamma^1$, D defaults and C gets $\gamma = \gamma^1$.

In region 3, D can pay P from the firm's assets, L. Again, if D pays P, then with probability ρ he keeps control over the assets, which is an efficient outcome. And with probability $1-\rho$, C gets control, in which case they may renegotiate. The renegotiation starts from the status quo: (a) the liquidation value of the remaining assets is $L - P$ (since D used the remainder to pay the P); and (b) D's private cash holdings are intact at R_1 +

[23]Notice that here we are appealing to the fact that the parties are risk neutral and the technology is linear, so that there are no gains from negotiating prior to the lottery.

T. From (2.1), we deduce that, provided D pays P, C's net payoff γ is

$$P \;+\; (1-\rho)\left(L \;-\; P \;+\; \alpha(R_1 + T)(1 - \frac{1}{s})\right) \;-\; T$$

$$= \rho P \;+\; (1-\rho)\left(M \;+\; \alpha T(1 - \frac{1}{s})\right) \;-\; T \;=\; \gamma^3, \text{ say.}$$

If $\gamma^3 > \gamma^1$, D defaults and C gets $\gamma = \gamma^1$.

Observe three things about C's net payoff γ (in all three regions). First, γ is never more than M (since $\gamma \le \gamma^1 \le M$). Second, γ is a function of the three variables M, s and V, and is nondecreasing in all of them.[24] Third, if $\alpha = 0$, γ is independent of V.

In effect, we can identify a state by the realization of the triplet $(M,s,\alpha V) \equiv z$, say. For a given terminal node (P,ρ) of the message game, denote C's payoff in state z by $\gamma\left((P,\rho)|z\right)$. We have shown that $\gamma\left((P,\rho)|z\right) \le M$ and that $\gamma\left((P,\rho)|z\right)$ is nondecreasing in $z = (M,s,\alpha V)$.

Consider two states $z = (M,s,\alpha V)$ and $z' = (M',s',\alpha V')$, for which $M' \le M$, $s' \le s$, and $\alpha V' \le \alpha V$.

For a message-game contract, suppose C and D play strategies $[\sigma_C,\sigma_D]$ and $[\sigma'_C,\sigma'_D]$ in states z and z' respectively. And suppose the mechanism specifies respective (P,ρ) pairs $(P[\sigma_C,\sigma_D],\rho[\sigma_C,\sigma_D])$ and $(P[\sigma'_C,\sigma'_D],\rho[\sigma'_C,\sigma'_D])$. Now since, in state z, C prefers σ_C to σ'_C,

$$\gamma\left((P[\sigma_C,\sigma_D],\rho[\sigma_C,\sigma_D])|z\right) \;\ge\; \gamma\left((P[\sigma'_C,\sigma_D],\rho[\sigma'_C,\sigma_D])|z\right). \qquad \text{(i)}$$

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[24] Clearly γ is nondecreasing in M, s and V within each of the three regions. Also, there are no discontinuities in γ across the boundaries of the regions: at the boundary of regions 1 and 2, $\gamma^1 \le \gamma^2$; and at the boundary of regions 2 and 3, $\gamma^2 = \gamma^3$.

Equally, in state z', D prefers σ'_D to σ_D. Remembering that all outcomes are bilaterally efficient, we may view this in terms of C's payoff:

$$\gamma\Big((P[\sigma'_C,\sigma'_D],\rho[\sigma'_C,\sigma'_D])\,|\,z'\Big) \;\le\; \gamma\Big((P[\sigma'_C,\sigma_D],\rho[\sigma'_C,\sigma_D])\,|\,z'\Big). \qquad \text{(ii)}$$

Finally, since $\gamma\Big((P,\rho)\,|\,z\Big)$ is nondecreasing in z,

$$\gamma\Big((P[\sigma'_C,\sigma_D],\rho[\sigma'_C,\sigma_D])\,|\,z\Big) \;\ge\; \gamma\Big((P[\sigma'_C,\sigma_D],\rho[\sigma'_C,\sigma_D])\,|\,z'\Big). \qquad \text{(iii)}$$

The LHS's of (i) and (ii) are C's equilibrium net payoffs in states z and z' respectively. Combining (i), (ii) and (iii) we obtain (4.3) and (4.4) in Lemma 2.

(4.2) is an immediate consequence of the fact that $\gamma\Big((P,\rho)\,|\,z\Big) \le M$.

Q.E.D.

Proof of Proposition 4

In the light of (4.4), we may identify a state by $(M,s,\alpha V)$. In this proof, it helps to write C's payoff as $g(M,s,\alpha V)$, rather than $g(M,s,V)$.

Take any $g^0(.,.,.)$ satisfying the three constraints:

$$E[g(M,s,\alpha V)] \;\ge\; I - w \qquad \text{(i)}$$

$$g(M,s,\alpha V) \;\le\; M \qquad \text{(ii)}$$

and $\quad g(M,s,\alpha V)$ is nondecreasing in M, s and αV. $\qquad$ (iii)

We proceed in two steps. First, we "flatten" g^0 in the s-αV plane. For each M, consider

$$g^1(M) \equiv E[g^0(M,s,\alpha V)|M]. \qquad \text{(iv)}$$

$g^1(m)$ obviously continues to satisfy (ii) and, by construction, satisfies (i). $g^1(M)$ is nondecreasing in M, thanks to affiliation.[25] Moreover, the minimand in (4.5) has (weakly) decreased. To see this, use the law of iterated expectations:

$$\begin{aligned} &E[s(g^0(M,s,\alpha V) - g^1(M))] \\ &\qquad = E[E[s(g^0(M,s,\alpha V) - g^1(M))|M]] \\ &\qquad \geq E[E[s|M]E[(g^0(M,s,\alpha V) - g^1(M))|M]] \\ &\qquad = 0 \text{ by (iv)} \end{aligned}$$

-- where the inequality follows from affiliation.[26]

The second step in the proof is to replace $g^1(M)$ by

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[25]See Theorem 23(iii) in Milgrom-Weber (1982), with their $Z \equiv (M,s,\alpha V)$, their $g(Z) \equiv g^0(M,s,\alpha V)$, and, for $M_1 > M_2$, their sublattice $S \equiv A_1 \cup A_2$ where $A_j = \{(M,s,\alpha V)|M = M_j\}$, $j = 1,2$. Taking $A = A_1$ and $\bar{A} = A_2$, we find $g^1(M_1) \geq g^1(M_2)$.

[26]See Theorem 23(ii) of Milgrom-Weber (1982), with their $Z \equiv (M,s,\alpha V)$, their $g(Z) \equiv s$, and their $h(Z) \equiv g^0(M,s,\alpha V) - g^1(M)$, conditioning on the sublattice $S = \{(M,s,\alpha V)|M\}$. On this sublattice, both their g and h are nondecreasing functions.

$$g^2(M) \equiv \min\{M,P\},$$

where P solves $E[\min\{M,P\}] = I - w$. This implies

$$E[g^1(M)] \geq I - w = E[g^2(M)]. \qquad \text{(v)}$$

$g^2(M)$ obviously satisfies (ii) and (iii) and, by construction, satisfies (i). To confirm that the minimand in (4.5) has (weakly) decreased, suppose M takes the J values $M_1 > \ldots > M_j > \ldots > M_J$. For $1 \leq j \leq J$, let $\pi_j(s)$ be the probability that $M = M_j$ conditional on s. By inspection, there exists some j^*, where $1 \leq j^* \leq J$, such that

$$g^1(M_j) > g^2(M_j) \qquad \text{for } 1 \leq j \leq j^*$$

$$\text{and} \quad g^1(M_j) \leq g^2(M_j) \qquad \text{for } j^*+1 \leq j \leq J.$$

Define

$$\Delta(s) \equiv E[(g^1(M) - g^2(M))|s].$$

Then for $s^+ > s^-$,

$$
\begin{aligned}
\Delta(s^+) &= \sum_{j=1}^{J} \pi_j(s^+)[g^1(M_j) - g^2(M_j)] \\
&= \sum_{j=1}^{J} \pi_j(s^-) \left(\frac{\pi_j(s^+)}{\pi_j(s^-)} \right) [g^1(M_j) - g^2(M_j)] \\
&\geq \left(\frac{\pi_{j*}(s^+)}{\pi_{j*}(s^-)} \right) \sum_{j=1}^{J} \pi_j(s^-) [g^1(M_j) - g^2(M_j)] \\
&= \left(\frac{\pi_{j*}(s^+)}{\pi_{j*}(s^-)} \right) \Delta(s^-)
\end{aligned}
$$

-- where the inequality follows from affiliation.[27] Hence $\Delta(s^-) \geq 0$ implies $\Delta(s^+) \geq 0$. That is, $\Delta(s)$ exhibits single crossing: there exists some s^* such that

$$(s - s^*)[\Delta(s) - \Delta(s^*)] \geq 0 \quad \text{for all } s.$$

Taking expectations and applying the law of iterated expectations,

$$E[s(g^1(M) - g^2(M))] \geq s^*E[g^1(M) - g^2(M)],$$

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[27] A direct consequence of Theorem 24 of Milgrom-Weber (1982) is that

$$\frac{\pi_j(s^+)}{\pi_j(s^-)}$$

is nonincreasing in j.

which is nonnegative by (v). Thus the minimand in (4.5) has (weakly) decreased.

Q.E.D.

Proof of Proposition 5

Here we shall first prove that, without any distributional assumptions, c(I) is increasing and piecewise linear in the interval $w \leq I \leq w/(1 - Em)$, and that the slope of c(I) equals Es for $w \leq I < w/(1 - \underline{m})$. Next, we will prove that if $E[s|m]$ is nondecreasing in m, c(I) is convex. The rest of the Proposition then follows directly from (5.4).

Let m take the values $m_1 > \ldots > m_j > \ldots > m_J$, with associated probabilities $\pi_j > 0$, $j = 1,\ldots,J$. For $j = 1,\ldots,J$, define

$$\mu_j \equiv 1 - m_j \sum_{k=1}^{j} \pi_k - \sum_{k=j+1}^{J} \pi_k m_k .$$

Now $\mu_1 < \ldots < \mu_j < \ldots < \mu_J$. Note that $\mu_J = 1 - m_J$; and $\mu_1 = 1 - Em$, which is strictly positive by assumption. For notational convenience, let $m_{J+1} = 0$ and $\mu_{J+1} = 1$.

We partition $[w, \frac{w}{\mu_1}]$ into J regions $R_J \cup \ldots \cup R_j \cup \ldots \cup R_1$ where

$$R_j \equiv [\frac{w}{\mu_{j+1}}, \frac{w}{\mu_j}] \qquad j = 1,\ldots,J.$$

The regions R_j, $1 \leq j \leq J$, are defined so that for $I \in R_j$ the $P = P(I)$, say, that solves $E[\min\{P, Im\}] = I - w$ lies between Im_{j+1} and Im_j. The slope of P(I) in the interior of region R_j is

$$\beta_j = \frac{1 - \sum_{k=j+1}^{J} \pi_k m_k}{\sum_{k=1}^{j} \pi_k} \qquad 1 \le j \le J.$$

Notice that

$$\beta_j - m_j = \frac{\mu_j}{\sum_{k=1}^{j} \pi_k} > 0. \tag{i}$$

And, for future reference, observe that, for $2 \le j \le J$,

$$(\beta_{j-1} - \beta_j) \sum_{k=1}^{j-1} \pi_k - \pi_j(\beta_j - m_j) = 0, \tag{ii}$$

which, from (i), implies that $\beta_{j-1} > \beta_j$.

Now for I in the interior of region R_j, $1 \le j \le J$, the slope of c(I) equals

$$\beta_j \sum_{k=1}^{j} \pi_k s_k + \sum_{k=j+1}^{J} \pi_k s_k m_k \equiv \phi_j, \text{ say,}$$

where $s_j \equiv E[s|m_j]$. Each ϕ_j is a positive constant for $j = 1,\ldots,J$. Hence, since c(I) is continuous across the boundaries of the regions, we deduce that c(I) is increasing and piece-wise linear in $w \le I \le \frac{w}{\mu_1}$. If $m_J = \underline{m} > 0$, then in the interval $[w, \frac{w}{1-\underline{m}})$ the slope of c(I), ϕ_J, equals $\beta_J \sum_{k=1}^{J} \pi_k s_k = Es$.

If $E[s|m]$ is nondecreasing in m then $s_1 \geq \ldots \geq s_J$. Now for $2 \leq j \leq J$,

$$\phi_{j-1} - \phi_j = (\beta_{j-1} - \beta_j) \sum_{k=1}^{j-1} \pi_k s_k - \pi_j s_j (\beta_j - m_j). \qquad \text{(iii)}$$

But since $s_1 \geq \ldots \geq s_k \geq \ldots \geq s_j$, and $\beta_{j-1} > \beta_j$, the RHS of (iii) is no less than s_j times the LHS of (ii). That is, $\phi_{j-1} \geq \phi_j$ for all $2 \leq j \leq J$, and c(I) is convex.

Q.E.D.

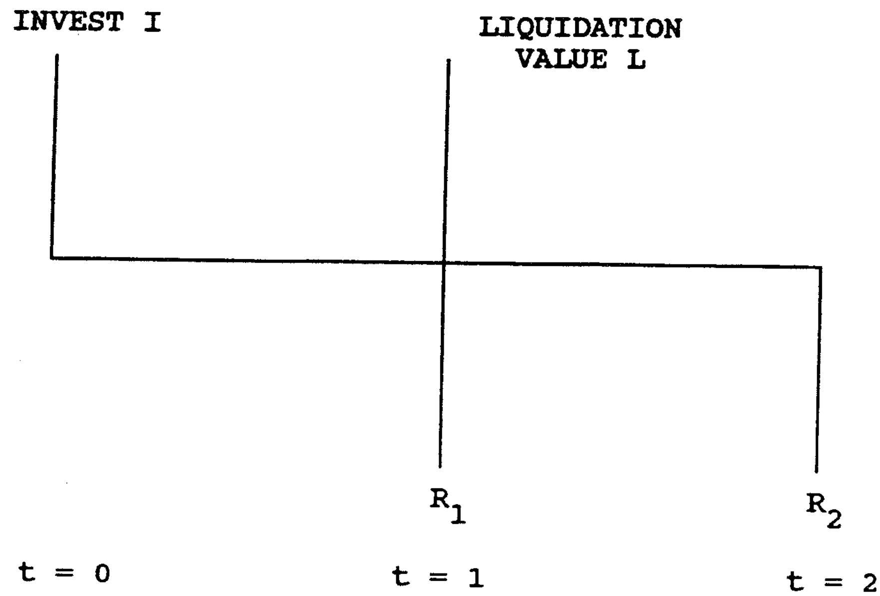

Figure 1

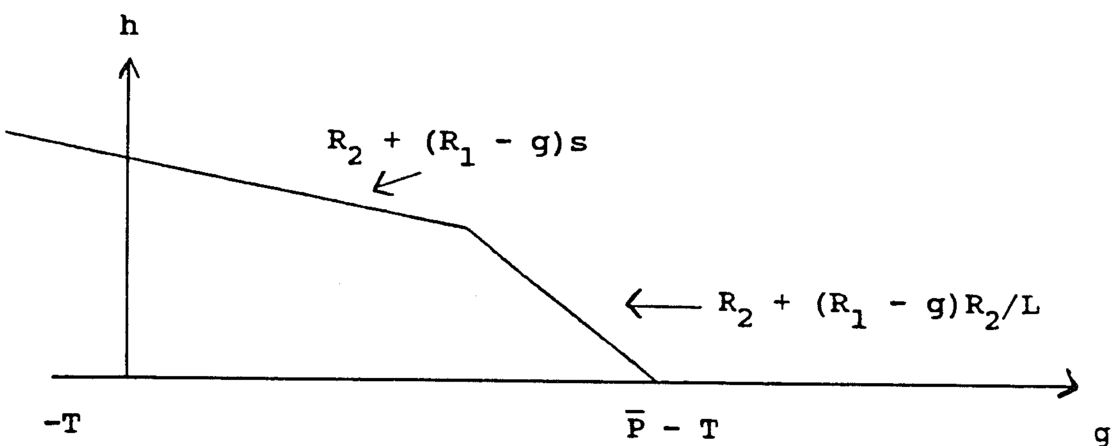

Figure 2

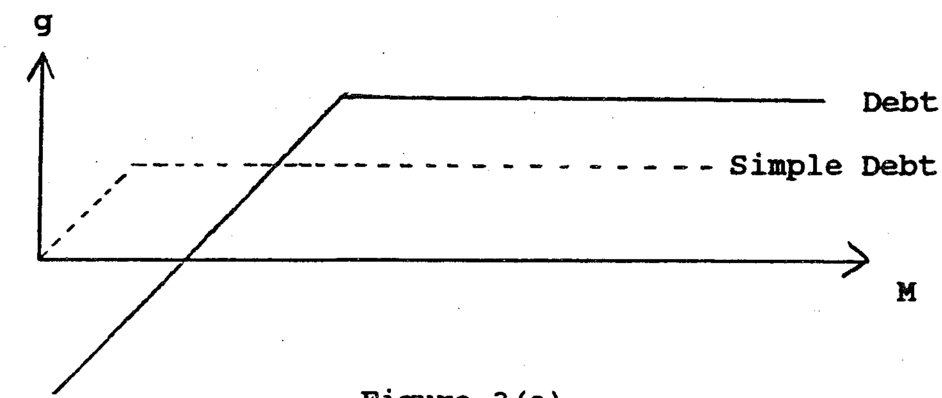

Figure 3(a)

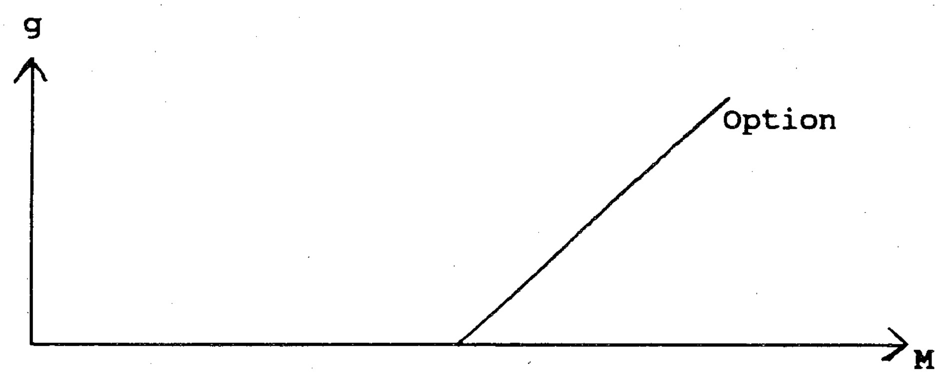

Figure 3(b)