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On the Representativeness of Voter Turnout*

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Abstract

Prominent theory research on voting uses models in which expected pivotality drives voters' turnout decisions and hence determines voting outcomes. It is recognized, however, that such work is at odds with Downs's paradox: in practice, many individuals turn out for reasons unrelated to pivotality, and their votes overwhelm the forces analyzed in pivotality-based models. Accordingly, we examine a complementary model of large-N elections at the opposite end of the spectrum, where pivotality effects vanish and turnout is driven entirely by individuals' direct costs and benefits from the act of voting itself. Under certain conditions, the level of turnout is irrelevant to representativeness and thus to voting outcomes. Under others, "anything is possible"; starting with any given distribution of preferences in the underlying population, there can arise any other distribution of preferences in the turnout set and thus any outcome within the range of the voting mechanism. Particular skews in terms of representativeness are characterized. The introduction of noise in the relationship between underlying preferences and individuals' direct costs and benefits from voting produces, in the limit, fully representative turnout. To illustrate the potential disconnect between the level of turnout (a focus of much empirical literature) and representativeness, we present a simple example in which, as noise increases, the turnout level monotonically falls yet representativeness monotonically rises.

JEL Codes: D71, D72

Keywords: voting; voter turnout; paradox of voting; pivotality; elections

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1 Introduction

Endogenous voter turnout and consequent voting outcomes are a focus of theoretical research on voting behavior. Game-theoretic analysis has been used to examine a variety of models in which turnout is driven by pivotality. For example, Ledyard (1984), Myerson (2000), and Krishna and Morgan (2015) consider models in which more intense preferences generate greater turnout, in light of voter's independently distributed costs, so as to produce utilitarian outcomes in the limit. Pursuing a different angle, Feddersen and Pesendorfer (1996, 1999) and McMurray (2013) show how, when voting is costless, endogenous turnout can efficiently aggregate voters' information as a consequence of strategic abstention by less-informed voters. See also Kim and Fey (2007) offering an extension under which aggregation may not be efficient in equilibrium and Martinelli (2006) examining costly information acquisition.

It is understood, however, that the pivotality-focused literature is in tension with an older, simpler line of work associated with Downs (1957). See, for example, the surveys by Feddersen (2004) and Mueller (2003). Because voters incur a positive cost, C, and the pivotality probability is extremely small in actual elections, there arises a paradox: voting is ex ante irrational for almost everyone yet turnout is substantial.[1] Theoretical and empirical research supports the conjecture that pivotality probabilities are indeed quite small in large-N elections with notable turnout. To explain nontrivial turnout, Riker and Ordeshook (1968) introduce D, the utility from the act of voting per se (see also Harsanyi (1969)). But once significant numbers of individuals turn out because $D > C$, the pivotality-driven effects identified in the game-theoretic literatures become vanishingly small.[2]

We undertake a positive, theoretical analysis of the determinants of voter turnout in large-N elections with a focus on the representativeness of the turnout set. In the spirit of work inspired by Downs (1957), we offer a perspective complementary to that in much of the modern theoretical literature by investigating what may be viewed as an opposite setting. Rather than taking C to be 0 or focusing on neighborhoods of 0 (when taking limits), while setting D to the side—which makes pivotality central and (in some models) turnout negligible—we instead employ a model with a continuum of individuals, where $D > C$ on a set of positive measure, so that turnout is infinite and therefore the probability that any vote is pivotal is literally 0. We explore how different relationships among the population distribution of preferences, C, and D map to turnout sets, with an emphasis on

[1] A related paradox, with similar suggested resolutions, concerns why voters (and even many nonvoters) undertake substantial information collection efforts. See, for example, Fiorina (1990).

[2] See also Palfrey and Rosenthal (1985), who conclude by noting that the introduction of even a small degree of uncertainty in their model produces a qualitatively different result, wherein only individuals with $D > C$ turn out.

its representativeness. Although not the focus, our results have straightforward implications for voting outcomes and provide insights into candidate positioning.

Section 2 reviews highlights of the literature on the probability of pivotality in large-N elections and relates our enterprise to existing theory. Section 3 presents our analysis. Like prior research, we seek to understand the composition of the turnout set, which is a selected sample that is not in general representative of the population as a whole. In our model, turnout is determined by the joint density function that characterizes the relationship between individuals' preferences and their Cs and Ds. Our results concerning the representativeness of turnout vary widely, depending on this density function, and they are qualitatively different from those in existing literatures.

Our first two propositions offer simple characterizations of polar cases. When preferences are distributed independently of C and D, the turnout set is representative: that is, the distribution of preferences in the turnout set is the same as that in the underlying population, and this is so no matter how low turnout is (as long as it is of positive measure). For any scale-invariant voting mechanism, such as majority rule or proportionality, the outcome is accordingly the same as that with full turnout.

By contrast, when no restrictions are placed on the relationship between preferences, C, and D, anything is possible. Specifically, starting with any given underlying distribution of preferences (with positive density on a compact domain), it is possible that the turnout set will have any other distribution of preferences. (For example, true preferences could be skewed extremely to the left with the resulting turnout set just as skewed to the right, or vice versa.) An implication is that, for any given true preferences, literally any outcome within the range of the voting mechanism can arise.

Our third and fourth propositions present the implications of particular interrelationships between preferences, C, and D. When the joint distribution is (approximately) deterministic—that is, when individuals with a given preference tend to have almost the same Cs and Ds—it is straightforward to characterize the turnout set. To take a special case, if the relationship between $D - C$ and preferences is linear (and such that a positive mass, but not everyone, turns out), then there is a simple skew in the turnout set and, accordingly, in the outcomes of simple voting schemes such as majority rule.

We then introduce noise: sequences of mean-preserving spreads of the distributions for C and D, conditional on preferences. In the limit, the turnout set is representative and the turnout level is 50%. We further demonstrate that there exist simple cases in which increasing noise causes the turnout set to become monotonically more representative (the median voter in the turnout set monotonically approaches the median in the population) and the turnout level to monotonically approach 50%. An immediate implication is that higher turnout does not go hand in hand with a more representative turnout set: here, whenever turnout is initially above 50%, increasing noise monotonically *raises* representativeness while monotonically *reducing* turnout.

Section 4 discusses how our results cast much prior literature in a new light—and suggests some reorientation of theoretical and empirical research on voting. Specifically, there is a need for greater focus on how C and D may systematically vary with underlying preferences—currently a feature of certain strands of empirical work on marginal voters, although without the structure provided by our analysis. As an illustration from the theoretical literature, we briefly return to the costly voting model of Ledyard (1984) and others. We explain how, on one hand, allowing a positive D renders much of the analysis moot. Even so, we find that putting certain structure on the joint distribution (allowing the magnitude of D to be correlated with preference intensity) can generate results in the spirit of those in the original papers, although other seemingly plausible assumptions produce quite different outcomes. We also revisit empirical work by Funk (2010) on mail-in voting that reveals competing effects on the *level* of turnout (it reduces Cs but also Ds), suggesting that such reforms may be unimportant. We consider instead the implications of these two distinct forces for the *composition* of turnout, which could significantly influence voting outcomes through both channels. More broadly, we explain how our emphasis on the turnout set illuminates a number of additional subjects, from endogenous candidate positioning to the assessment of a variety of voting reforms.

Before proceeding, we offer the important caveat that our analysis is entirely positive. A more representative turnout set need not generate socially superior outcomes for such standard reasons as that majority rule does not weight preferences by intensity, underlying preferences may not be equally well-informed, and they are subject to manipulation via electioneering. Nevertheless, understanding the determinants of the turnout set is fundamental in analyzing a range of relevant behaviors and policies, particularly those aimed at influencing voter turnout—including activities of political campaigns and voting reforms aimed at enhancing or suppressing the turnout of individuals likely to have particular preferences.

2 Literature

The voting literature is vast. Here we present some of the work related to pivotality, with additional references more pertinent to the implications of our analysis deferred to section 4. We begin with analyses of the probability of pivotality in large-N elections, which provide a perspective on important lines of the theoretical literature and a motivation for our investigation.

Chamberlain and Rothschild (1981) examine a model in which there is a majority vote between two alternatives, voters' preferences are drawn from a binomial distribution, and the pertinent probability for that distribution is itself drawn from another distribution. They derive that the probability of an individual being pivotal is on the order of $1/N$ in large-N elections.[3]

Gelman, Silver, and Edlin (2012) offer an empirical analysis of the 2008 presidential election and find that the probability of a vote being pivotal (taking into account complexities due to the Electoral College) was approximately 1 in 60 million. In a number of states, the probability was less than 1 in a billion; at the other end of the spectrum, in a handful of states, the probability was around 1 in 10 million. Although the pivotality probability varied by orders of magnitude across the states, the turnout differentials did not reflect even a sliver of that difference.[4]

Mulligan and Hunger (2003) assembled a substantial historical database of House elections and state legislative races, both of which have vastly smaller electorates than the nation as a whole. The pivotality probability was roughly 1 in 89,000 for the former and 1 in 15,000 for the latter.[5]

Enos and Fowler (2014, 310) aggressively advanced a general conclusion about the exceedingly low probabilities of being pivotal in large-N elections:[6]

> This model [in which individuals vote if and only if $PB - C + D > 0$] provides the basis for much of the current understanding of the decision to vote. However,

[3]Specifically, in the limit as $N \to \infty$ the probability of being pivotal is $1/2N$ times the magnitude of the (assumed to be continuous) density of the distribution of true probabilities, evaluated at 0.5.

[4]Some empirical work suggests that closeness influences turnout to some extent (Geys 2006), but even those modest effects (relative to orders-of-magnitude differences in the pivotality probability) may be explained by the fact that closer races generate greater publicity, higher campaign expenditures, and more aggressive get-out-the-vote campaigns (Cox and Munger (1989) and Enos and Fowler (2018)). And surveys suggesting that voters who turn out have higher estimates of the probability of being pivotal may reflect cognitive dissonance in providing ex post rationalizations rather than actual, behavior-inducing core beliefs (Enos and Fowler 2014). Field experiments by Gerber et al. (2017) indicate that perceived closeness has essentially no effect on turnout.

[5]Their results are in an important respect the most convincing because they impute the likelihood of a tie (or one vote from a tie) from the actual distribution of the vote differential in all of the very close contests that occurred (in a sample of roughly 17,000 House elections and 40,000 state contests). Hence, they rely less on the sorts of parametric assumptions that are required for most other empirical or theoretical exercises.

[6]This literature understates pivotality probabilities by focusing on single contests in broader elections, but scaling the probabilities by, say, a factor of ten does not significantly alter the conclusion. In addition, Schwartz (1987) argues that individuals may vote because outcomes in their precinct may influence local governments' distribution of benefits, which suggests (contrary to observed behavior) that local elections should generate greater turnout than national ones. Similarly, it is notable that, even in jurisdictions where one party is dominant, primaries usually have much lower turnouts (despite the implied higher pivotality probabilities) than those in general elections; this phenomenon suggests that there are social determinants of individuals Ds of the sort we elaborate in section 4.

> P is infinitesimal for any large election ..., so changes to P should have little effect on turnout. As Schwartz (1987) aptly points out, "Saying that closeness increases the probability of being pivotal is like saying that tall men are more likely than short men to bump their heads on the moon." Gerber, Green and Larimer (2008), echoing Schwartz, note that "Because the probability of casting a decisive vote in an election is typically infinitesimal, the calculus of voting boils down to the relative weight of C and D."

By contrast, much of the modern theory research on voting omits D and either takes C to equal 0 or examines (in the limit as the population grows) C in neighborhoods of 0.[7] Consider Ledyard (1984), which has been extended by Myerson (2000) and Krishna and Morgan (2015), among others.[8] The central model posits that individuals vote when their utility from being pivotal exceeds their cost of voting, C. An equilibrium with positive (although extremely small) turnout arises because, if almost no one turned out, the pivotality probability would be large (generating a utility from pivotality exceeding C for those with sufficiently low C), whereas if many turned out, the probability would be too small to sustain that turnout level. The key substantive result is that the outcome is utilitarian because the turnout set accurately reflects preference intensity, whereas majority vote with full turnout would not.

The theoretical literature just discussed assumes that preferences and individuals' Cs are distributed independently and, moreover, that the density of the distribution of C is positive on $[0, 1]$ (with no point mass at 0). The intuition can be seen by recognizing that, when one examines neighborhoods of 0, the density will be approximately uniform. In a large but finite population (individuals with given preferences have their Cs drawn from the aforementioned distribution), the number who turn out for any given intensity is proportional to that intensity. This can be understood by positing an equilibrium P (which, as N goes to infinity, we can take to be small) and noting that the fraction of voters who have their randomly chosen C fall below PB will (in the limit) be proportional to B, the individual's benefit from swinging the outcome.

Clearly, if one introduces even a very small D into these models in which all those who vote have Cs in the neighborhood of 0, a notable fraction of individuals will choose to vote

[7] A sizable portion of a larger voting literature likewise emphasizes pivotality in some fashion. Enos and Fowler (2014, 310) reviewed "all articles published on voter turnout since 1980 that appear in five leading political science journals. Of the 70 articles ... that addressed the causes of voter turnout, 41 made a clear appeal to the importance of pivotality or electoral competition."

[8] In Ledyard's paper that launched this line of work, candidates endogenously choose positions, and equilibria can have the form that literally no one votes because both candidates choose identical positions at the electorate's preferred point (in the sense discussed in the text to follow).

regardless of pivotality—indeed, an infinite number in the limit. The probability of pivotality approaches 0 even more rapidly, implying that preference-intensity-based turnout in these models becomes a negligible fraction of all voters.

To take another example, Feddersen and Pesendorfer (1996, 1999), Kim and Fey (2007), and McMurray (2013) explore when endogenous turnout can efficiently aggregate voters' information as a consequence of strategic abstention by less-informed voters. In this model, $C = 0$ and $D = 0$ for everyone. Hence, voting is motivated entirely by pivotality, and individuals wish to vote only if their information indicates that they would gain from tipping the election when their vote is decisive. If C were instead positive for almost everyone, few would vote, as noted by Feddersen (2004).[9] Moreover, as McMurray (2015) suggests, if many have a positive D as well, large numbers of uninformed individuals will vote, which renders the pivotality probability negligible and the information aggregation effect nil.

In the foregoing literatures and others, the analysis and results would be undermined in models that include both C and D, thereby yielding substantial turnout and, accordingly, trivial probabilities of pivotality—consistent with the aforementioned literature on that subject. Pivotality effects are not merely supplemented by others but essentially vanish. This is our point of departure: our model takes the pivotality probability to be 0 and focuses instead on characterization of the turnout set when voters are those for whom $D - C > 0$. In section 4, we return to the existing literature and explain how, under certain assumptions, results similar to those obtained in the prior literature can be resurrected in our framework, albeit through a different mechanism.

3 Model and Analysis

There is a compact set X of outcomes. Each agent i has a type consisting of a *preferred outcome* $x^i \in X$, a personal *cost* $C^i \geq 0$ of voting, and a personal *benefit* $D^i \geq 0$ from the act of voting itself.

The distribution of agent types in the population has density function $f(x^i, D^i, C^i)$, which we assume to be continuous. Here and throughout, we abuse notation by writing Π_f for an operator to indicate corresponding marginal and conditional densities.[10] Then, we denote

[9] Compounding the conundrum is that the costs and cognitive challenges individuals face in determining whether or how to vote in these information aggregation models may be far greater than the costs of simply becoming informed to begin with and in any event likely outweigh their povitality benefit from voting.

[10] We use the somewhat unorthodox convention of writing "$x^i =$" in $\Pi_f(x^i = x)$ and similar expressions, to make the notation more intuitively comparable with probabilities.

by

$$\tilde{f}(x) \equiv \Pi_f(x^i = x)$$
$$= \int_0^\infty \int_0^\infty f(x, D, C)\, dCdD$$

the (unconditional) distribution of agent preferences in the population, which we assume throughout to be strictly positive.

All agents i for whom $D^i - C^i > 0$ *turn out* (that is, they show up) to vote;[11] we assume that that $\text{Prob}_f[D^i - C^i > 0] > 0$ (i.e., a positive measure of agents turn out). Then, the *realized distribution of preferences of agents that turn out*, which has density

$$\widetilde{t_f}(x) \equiv \Pi_f(x^i = x \mid D^i - C^i > 0)$$
$$= \frac{\int_0^\infty \int_0^D f(x, D, C)\, dCdD}{\int_X \left(\int_0^\infty \int_0^D f(\hat{x}, D, C)\, dCdD\right) d\hat{x}},$$

is well-defined and non-degenerate. We say that the turnout is *representative* if $\widetilde{t_f}(x) = \tilde{f}(x)$ for all x.

The *voting mechanism* $\mathcal{V}$ can be taken to be a function that maps the set of agents that turn out into a social outcome in X. We assume that this function is scale-invariant.

Assumption 1. The voting mechanism $\mathcal{V}$ is *scale-invariant*, i.e., $\mathcal{V}$ chooses an outcome $\mathcal{V}(\widetilde{t_f})$ that depends only on the realized vote distribution.

Assumption 1 states that the outcome of the mechanism depends only on the distribution of preferences of the voters who turn out. In particular, the chosen outcome does not depend on the mass of voters that shows up. Thus we immediately obtain our first result, *aggregate turnout irrelevance*: when the density of types, f, is suitably restricted, the outcome under realized turnout matches the outcome under full turnout, no matter how low aggregate turnout is.

Proposition 1. *If x^i is independent of (D^i, C^i) under f, then turnout is representative and $\mathcal{V}(\widetilde{t_f}) = \mathcal{V}(\tilde{f})$, that is, the outcome of the voting mechanism under the realized turnout is the same as the outcome that would be selected under full turnout.*

[11] Note that, by convention, we assume that agents who are indifferent regarding turnout (i.e., those agents i for whom $D^i - C^i = 0$) do *not* turn out; this does not affect the turnout distribution because f admits no atoms in $D^i - C^i$.

Proof. The representativeness of turnout is immediate by the independence assumption. Indeed, we have

$$\begin{aligned}
\widetilde{t_f}(x) &= \Pi_f(x^i = x \mid D^i - C^i > 0) \\
&= \frac{\Pi_f(x^i = x \mid D^i - C^i > 0) \cdot \mathrm{Prob}_f[D^i - C^i > 0]}{\mathrm{Prob}_f[D^i - C^i > 0]} \\
&= \frac{\Pi_f(x^i = x) \cdot \mathrm{Prob}_f[D^i - C^i > 0]}{\mathrm{Prob}_f[D^i - C^i > 0]} \\
&= \Pi_f(x^i = x) \\
&= \tilde{f}(x),
\end{aligned}$$

where the third equality follows from the independence assumption. Assumption 1 then implies that $\mathcal{V}(\widetilde{t_f}) = \mathcal{V}(\tilde{f})$. $\square$

We now show that for an unrestricted type density f, in an important sense *anything can happen* regarding voter turnout and voting outcomes.

Proposition 2. *Fix a mechanism $\mathcal{V}$, as well as the (unconditional) preference distribution in the population, $\tilde{f}$. For any alternative preference distribution $\tilde{g}$, there exists a type distribution f consistent with $\tilde{f}$ such that $\widetilde{t_f} = \tilde{g}$, which further implies that $\mathcal{V}(\widetilde{t_f}) = \mathcal{V}(\tilde{g})$.*

Proof. For any given $\tilde{f}$ and $\tilde{g}$, we construct the desired type distribution f explicitly. First, we let Z be any positive real number such that

$$\max_{x \in X} \left\{ \frac{\tilde{g}(x)}{Z \cdot \tilde{f}(x)} \right\} < 1.$$[12]

To construct a type distribution with the requisite properties—matching both the underlying preference distribution $\tilde{f}$ and the turnout distribution $\tilde{g}$—we proceed as follows: For each x, our constructed type distribution f will equal the targeted $\tilde{f}$ on a specified unit interval for values of $D^i - C^i$ and it will equal 0 otherwise, thereby replicating the underlying preference distribution. Moreover, the unit interval for each x will involve a $D^i - C^i$ that is positive (meaning that individuals will turn out) on a fraction of the associated unit interval proportional to the targeted $\tilde{g}$, thereby generating the requisite turnout distribution. Specifically,

[12]Such a number Z exists because both $\tilde{g}$ and $\tilde{f}$ are continuous and strictly positive, X is compact, and continuous functions on compact sets achieve their maxima. Indeed, the assumptions on $\tilde{g}$ and $\tilde{f}$ imply that $\frac{\tilde{g}(x)}{\tilde{f}(x)}$ is continuous on X as well; hence, it achieves some maximum Z' on X. Taking $Z > Z'$ suffices.

we let

$$f(x^i, D^i, C^i) = \begin{cases} \tilde{f}(x^i) & -\left(1 - \frac{\tilde{g}(x^i)}{Z\cdot\tilde{f}(x^i)}\right) \leq D^i - C^i \leq \frac{\tilde{g}(x^i)}{Z\cdot\tilde{f}(x^i)}, \\ 0 & \text{otherwise.} \end{cases}$$

Now, for each $x \in X$, we have

$$\tilde{f}(x) = \int_0^\infty \int_0^\infty f(x, D, C)\, dCdD,$$

as desired. Moreover, we have

$$\begin{aligned} \widetilde{t_f}(x) &= \Pi_f(x^i = x \mid D^i - C^i > 0) \\ &= \frac{\int_0^\infty \int_0^D f(x, D, C)\, dCdD}{\int_X \left(\int_0^\infty \int_0^D f(\hat{x}, D, C)\, dCdD\right) d\hat{x}} \\ &= \frac{\tilde{f}(x)\frac{\tilde{g}(x)}{Z\tilde{f}(x)}}{\int_X \tilde{f}(\hat{x})\frac{\tilde{g}(\hat{x})}{Z\tilde{f}(\hat{x})}\, d\hat{x}} \\ &= \frac{\tilde{g}(x)}{\int_X \tilde{g}(\hat{x})\, d\hat{x}} \\ &= \tilde{g}(x), \end{aligned}$$

as we have $\int_X \tilde{g}(\hat{x})\, d\hat{x} = 1$. □

For the remainder of our analysis, we present some limiting results that further characterize possible relationships between turnout and representativeness. In the first, where we take the variance of individuals' voting costs and benefits to 0, we find that the costs and benefits determine turnout for each x in a simple way. We let $\mu_f(x) \equiv \mathrm{E}_f[D^i - C^i \mid x^i = x]$ and $\nu_f(x) \equiv \mathrm{Var}_f[D^i - C^i \mid x^i = x]$ be the (assumed finite) mean and variance of the net payoffs to voting for agents who prefer outcome x. We denote by

$$\mathbb{1}_{\mu_f(x)>0} = \begin{cases} 1 & \mu_f(x) > 0 \\ 0 & \mu_f(x) < 0 \end{cases}$$

the indicator function for whether $\mu_f(x) > 0$; that is, $\mathbb{1}_{\mu_f(x)>0}$ indicates whether the mean of $D^i - C^i$ for agents of preference type x is above or below 0.

Let

$$\Sigma \equiv \int_X [\mathbb{1}_{\mu_f(x)>0} \cdot \tilde{f}(x)]\, dx$$

be the measure of outcomes x for which $\mu_f(x) > 0$. In the sequel, we use the technical assumption that $\{x \in X : \mu_{f_n}(x) = \mu_f(x) = 0\}$ has measure 0 (so that outcomes x with $\mu_f(x) = 0$ do not affect our integrals); moreover, we omit characterizations of the limit for any x in this set.

Proposition 3. *Suppose that f_n is a sequence of agent type distributions with $\tilde{f}_n(x) = \tilde{f}(x)$, $\mu_{f_n}(x) = \mu_f(x)$, and $\nu_{f_n}(x) \to 0$ for each $x \in X$. Then, as $n \to \infty$, the total measure of agents that turn out approaches Σ, and the realized distribution of preferences of agents that turn out, $\widetilde{t_{f_n}}(x)$, approaches*

$$\widetilde{h_f}(x) \equiv \frac{\mathbb{1}_{\mu_f(x)>0} \cdot \tilde{f}(x)}{\Sigma} = \begin{cases} \frac{\tilde{f}(x)}{\Sigma} & \mu_f(x) > 0 \\ 0 & \mu_f(x) < 0. \end{cases}$$

Proof. By Chebyshev's inequality, for any $x \in X$ for which $\mu_f(x) \neq 0$, we have

$$\mathrm{Prob}_{f_n}\left[|(D^i - C^i) - \mu_{f_n}(x)| \geq k \mid x^i = x\right] \leq \frac{\nu_{f_n}(x)}{k^2} \tag{1}$$

for any $k > 0$. By assumption, $\nu_{f_n}(x) \to 0$ as $n \to \infty$. Thus, fixing $x \in X$ with $\mu_{f_n}(x) \neq 0$ and taking $k < |\mu_{f_n}(x)|$, we have from (1) that as $n \to \infty$,

$$\mathrm{Prob}_{f_n}\left[|(D^i - C^i) - \mu_{f_n}(x)| \geq k \mid x^i = x\right] \to 0.$$

Therefore, when $\mu_{f_n}(x) < 0$, we have, as $n \to \infty$,

$$\mathrm{Prob}_{f_n}\left[D^i - C^i > 0 \mid x^i = x\right] \to 0.$$

Likewise, when $\mu_{f_n}(x) > 0$, as $n \to \infty$,

$$\mathrm{Prob}_{f_n}\left[D^i - C^i > 0 \mid x^i = x\right] \to 1.$$

Thus, as $n \to \infty$,

$$\mathrm{Prob}_{f_n}[D^i - C^i > 0 \mid x^i = x] \to \mathbb{1}_{\mu_{f_n}(x)>0}. \tag{2}$$

It then follows from (2) that, as $n \to \infty$, the total measure of agents who turn out,

$$\mathrm{Prob}_{f_n}[D^i - C^i > 0],$$

approaches

$$\int_X [\mathbb{1}_{\mu_{f_n}(x)>0} \cdot \tilde{f}(x)]\, dx = \Sigma. \tag{3}$$

Now, we compute that

$$\begin{aligned}
\widetilde{t_{f_n}}(x) &= \Pi_{f_n}(x^i = x \mid D^i - C^i > 0) \\
&= \frac{\Pi_{f_n}\left[x^i = x \mid D^i - C^i > 0\right] \cdot \mathrm{Prob}_{f_n}[D^i - C^i > 0]}{\mathrm{Prob}_{f_n}[D^i - C^i > 0]} \\
&= \frac{\mathrm{Prob}_{f_n}\left[D^i - C^i > 0 \mid x^i = x\right] \cdot \Pi_{f_n}(x^i = x)}{\mathrm{Prob}_{f_n}[D^i - C^i > 0]} \\
&\to \frac{\mathbb{1}_{\mu_f(x)>0} \cdot \tilde{f}(x)}{\Sigma} \\
&= \widetilde{h_f}(x),
\end{aligned}$$

as claimed. □

Let $\mathcal{V}^{\mathrm{med}}$ be a *median voter mechanism*—that is, $\mathcal{V}^{\mathrm{med}}(\widetilde{t_f}) = x^{\mathrm{med}}(\widetilde{t_f})$, where $x^{\mathrm{med}}(\widetilde{t_f})$ is the median x^i in the turnout set. It follows immediately from Proposition 3 that:

Corollary 1. *As $n \to \infty$, we have $\mathcal{V}^{\mathrm{med}}(\widetilde{t_{f_n}}) \to x^{med}(\widetilde{h_f})$.*

To give a simple, concrete illustration of a limiting turnout set of the sort characterized in Proposition 3, consider the following example.

Example 1. Suppose that $X = [\underline{x}, \overline{x}]$ for some $\underline{x}, \overline{x} \in \mathbb{R}$ (with $\underline{x} < \overline{x}$), with $\tilde{f}$ uniform. Suppose further that, conditional on x^i, $D^i - C^i$ is distributed $\sim \mathsf{N}(\mu_f(x^i), \nu_{f_n})$, that is, as a normal random variable with (fixed) mean $\mu_f(x^i)$ and (common) variance ν_{f_n}, which, as in the construction for Proposition 3, we take to be trending to 0. We furthermore assume that $\mu_f(x)$ is linear and upward-sloping in x, with $\mu_f(\underline{x}) < 0$ and $\mu_f(\overline{x}) > 0$. We let $x^{\mathrm{cut}} \equiv \mu_f^{-1}(0)$ be the point x at which the mean of $D^i - C^i$ is exactly 0, and let x_n^{med} be the preference of the median voter in the turnout set. The setting just described is pictured in Figure 1.

In examining Figure 1, we can see that the turnout set—and accordingly, x_n^{med}—will be skewed toward the right. Moreover, as Proposition 3 indicates, in the limit turnout would be determined entirely by the line itself: all those to the right of x^{cut} turn out; all those to the left do not. Given our assumed uniform (unconditional) distribution of x^i, x_n^{med} would be half way between x^{cut} and 1 in the limit. The configuration depicted in Figure 1—albeit more plausibly a shaded version indicating significant dispersion of the distributions of $D^i - C^i$ at

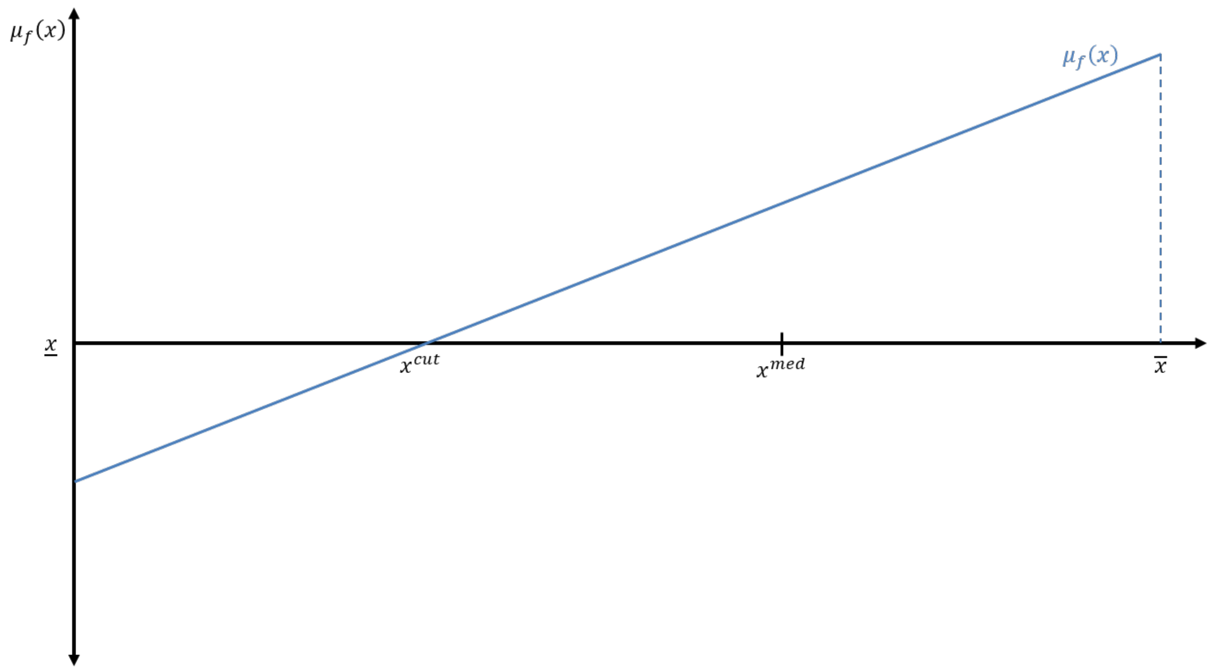

Figure 1: Linear Mean; $\nu_{f_n} \to 0$

each x^i—might arise, for example, if those agents i with higher x^i tend to have more group cohesion (and thus a higher D^i), or tend to have lower time costs of turning out (and thus a lower C^i).

We now introduce what may be regarded as a sort of inverse to Proposition 3, in which we take dispersion to increase without bound rather than vanish in the limit.[13] To do this, we now let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be sequences of random variables with the same strictly positive (but finite) variance. We suppose that the variables α_n are identically distributed and independent of each other, the D^i, the C^i, and the β_n; we likewise suppose that the variables β_n are identically distributed and independent of each other, the D^i, the C^i, and (by our earlier assumption) of the α_n. Moreover, we suppose that

$$\mathrm{E}[\alpha_n] = 0, \quad \mathrm{E}[\beta_n] = 0.$$

We let f_n be the distribution of agent types under which agents' costs and benefits of voting are given by $C^i + \alpha_1 + \cdots + \alpha_n$ and $D^i + \beta_1 + \cdots + \beta_n$, respectively. The distributions of costs and benefits under f_n are therefore mean-preserving spreads of the distributions of costs and benefits under f_{n-1}; we obtain $f_1, f_2, \ldots$ from f by iteratively spreading C^i and

[13] Our results in this setting bear a kinship to the phenomenon of attenuation bias due to classical measurement error.

D^i according to α and β, respectively.[14] By convention, we write $f_0 \equiv f$.

Proposition 4. *For any f, as $n \to \infty$,*

- *for each $x \in X$, the measure of agents i with $x^i = x$ who turn out approaches $\frac{1}{2}$,*
- *the overall measure of agents who turn out approaches $\frac{1}{2}$, and*
- *the realized distribution of preferences of agents that turn out, $\widetilde{t_{f_n}}$, approaches $\tilde{f}$.*

If $\mathcal{V}$ is continuous, then the outcome $\mathcal{V}(\widetilde{t_{f_n}})$ approaches the outcome that would be selected under full turnout, $\mathcal{V}(\tilde{f})$.

Proof. We denote $\varepsilon_n \equiv \beta_n - \alpha_n$. We note that the $\{\varepsilon_n\}_{n=1}^{\infty}$ have strictly positive (but finite) variance, which we denote by υ. Moreover, the $\{\varepsilon_n\}_{n=1}^{\infty}$ are independent and identically distributed, with $\mathrm{E}[\varepsilon_n \mid D^i - C^i] = 0$ by construction.

We let

$$\Delta_n^i \equiv (D^i + \beta_1 + \cdots + \beta_n) - (C^i + \alpha_1 + \cdots + \alpha_n) = (D^i - C^i) + \varepsilon_1 + \cdots + \varepsilon_n.$$

We have, as $n \to \infty$,

$$\frac{\varepsilon_1 + \cdots + \varepsilon_n}{\sqrt{\upsilon} \cdot \sqrt{n}} \xrightarrow{d} \mathsf{N}(0,1) \tag{4}$$

by the Central Limit Theorem. Meanwhile, as $n \to \infty$, we have

$$\frac{D^i - C^i}{\sqrt{\upsilon} \cdot \sqrt{n}} \xrightarrow{d} 0 \tag{5}$$

almost surely. Combining (4) and (5), we see that as $n \to \infty$,

$$\frac{(D^i - C^i) + (\varepsilon_1 + \cdots + \varepsilon_n)}{\sqrt{\upsilon} \cdot \sqrt{n}} \xrightarrow{d} \mathsf{N}(0,1).$$

[14] Most of the assumptions we use are for simplicity and are stronger than we need for the result. The result we present is unchanged if the mean-preserving spreads differ within each sequence, as long as there is a uniform and strictly positive lower bound on the variance of $\beta_n - \alpha_n$. Additionally, the result is unchanged if the mean-preserving spreads depend on x^i (again, given the required variance lower bound).

Hence, we have that

$$
\begin{aligned}
\text{Prob}[\Delta_n^i > 0] &= \text{Prob}\left[\frac{\Delta_n^i}{\sqrt{\upsilon}\cdot\sqrt{n}} > 0\right] \\
&= \text{Prob}\left[\frac{(D^i - C^i) + (\varepsilon_1 + \cdots + \varepsilon_n)}{\sqrt{\upsilon}\cdot\sqrt{n}} > 0\right] \\
&\to 1 - \Phi(0) \\
&= \frac{1}{2}, \qquad (6)
\end{aligned}
$$

where Φ is the cumulative distribution function of the standard Normal.[15]

It follows from (6) that, in the limit as $n \to \infty$, exactly half of the agents of each preference type x turn out. Indeed, the measure of agents of preference type x who turn out is

$$
\begin{aligned}
&\Pi_{f_n}(x^i = x \mid (D^i - C^i) + \varepsilon_1 + \cdots + \varepsilon_n > 0) \cdot \text{Prob}_{f_n}[(D^i - C^i) + \varepsilon_1 + \cdots + \varepsilon_n > 0] \\
&= \Pi_{f_n}(x^i = x \mid \Delta_n^i > 0) \cdot \text{Prob}_{f_n}[\Delta_n^i > 0] \\
&= \text{Prob}_{f_n}[\Delta_n^i > 0 \mid x^i = x] \cdot \Pi_{f_n}(x^i = x) \\
&\to \frac{1}{2} \cdot \Pi_f(x^i = x) \\
&= \frac{\tilde{f}(x)}{2}.
\end{aligned}
$$

The total measure of agents who turn out then approaches

$$\int_X \left(\frac{\tilde{f}(x)}{2}\right) dx = \frac{1}{2}.$$

[15] We thank Nate Eldredge for suggesting the core of this argument.

Moreover, we have $\widetilde{t_{f_n}}(x) \to \tilde{f}(x)$, as claimed:

$$\begin{aligned}
\widetilde{t_{f_n}}(x) &= \Pi_{f_n}(x^i = x \mid (D^i - C^i) + \varepsilon_1 + \cdots + \varepsilon_n > 0) \\
&= \Pi_{f_n}(x^i = x \mid \Delta_n^i > 0) \\
&= \frac{\Pi_{f_n}(x^i = x \mid \Delta_n^i > 0) \cdot \mathrm{Prob}_{f_n}[\Delta_n^i > 0]}{\mathrm{Prob}_{f_n}[\Delta_n^i > 0]} \\
&= \frac{\mathrm{Prob}_{f_n}[\Delta_n^i > 0 \mid x^i = x] \cdot \Pi_{f_n}(x^i = x)}{\mathrm{Prob}_{f_n}[\Delta_n^i > 0]} \\
&\to \frac{\frac{1}{2} \cdot \Pi_f(x^i = x)}{\frac{1}{2}} \\
&= \tilde{f}(x).
\end{aligned}$$

The result on $\mathcal{V}(\widetilde{t_{f_n}})$ then follows immediately by the continuity of $\mathcal{V}$. □

To illustrate Proposition 4, we now return to a setting with a uniform distribution of x and $\mu_f(x)$ rising linearly, akin to the setting we considered in Example 1. This example will also present a case in which the convergence is monotonic; that is, turnout for every x approaches $\frac{1}{2}$ monotonically, meaning that overall turnout does as well. Moreover, we will show that the median voter in the turnout set monotonically approaches the true median preference in the population.

Example 2. Suppose that, like in Example 1, $X = [\underline{x}, \overline{x}]$ for some $\underline{x}, \overline{x} \in \mathbb{R}$ (with $\underline{x} < \overline{x}$), with $\tilde{f}$ uniform, and suppose that, conditional on x^i, $D^i - C^i$ is distributed $\sim \mathsf{N}(\mu_f(x^i), \nu_{f_n})$, that is, as a normal random variable with (fixed) mean $\mu_f(x^i)$ and common variance ν_{f_n}. But now, as in the construction for Proposition 4, we take ν_{f_n} to *grow* in n. As before, we assume that $\mu_f(x)$ is linear and upward-sloping in x, with $\mu_f(\underline{x}) < 0$ and $\mu_f(\overline{x}) > 0$. We let $x^{\mathrm{cut}} \equiv \mu_f^{-1}(0)$ be the point at which the mean of $D^i - C^i$ is exactly 0. We now let $d \equiv |x^{\mathrm{cut}} - \underline{x}|$ and assume that $x^{\mathrm{cut}} + d < \overline{x}$. For notational simplicity, we abuse notation slightly by writing $\mu^x \equiv \mu_f(x)$ and $\nu_n \equiv \nu_{f_n}$. The setting just described is pictured in Figure 2.

To prove monotonicity in addition to convergence to $\frac{1}{2}$, we examine how turnout changes in different segments of the preference distribution. First, we observe that as $\mu^{x^{\mathrm{cut}}} = 0$, the linearity of μ^x and symmetry of the normal distributions $\mathsf{N}(\mu^x, \nu_n)$ (along with the uniformity of $\tilde{f}$) imply that, for any $x \in [\underline{x}, x^{\mathrm{cut}})$, the mass of agents i with $x^i = x$ and $D^i - C^i > 0$ is precisely equal to the mass of agents i with $D^i - C^i \le 0$ and

$$x^i = [x^{\mathrm{cut}} + |x^{\mathrm{cut}} - x|] \in (x^{\mathrm{cut}}, x^{\mathrm{cut}} + d],$$

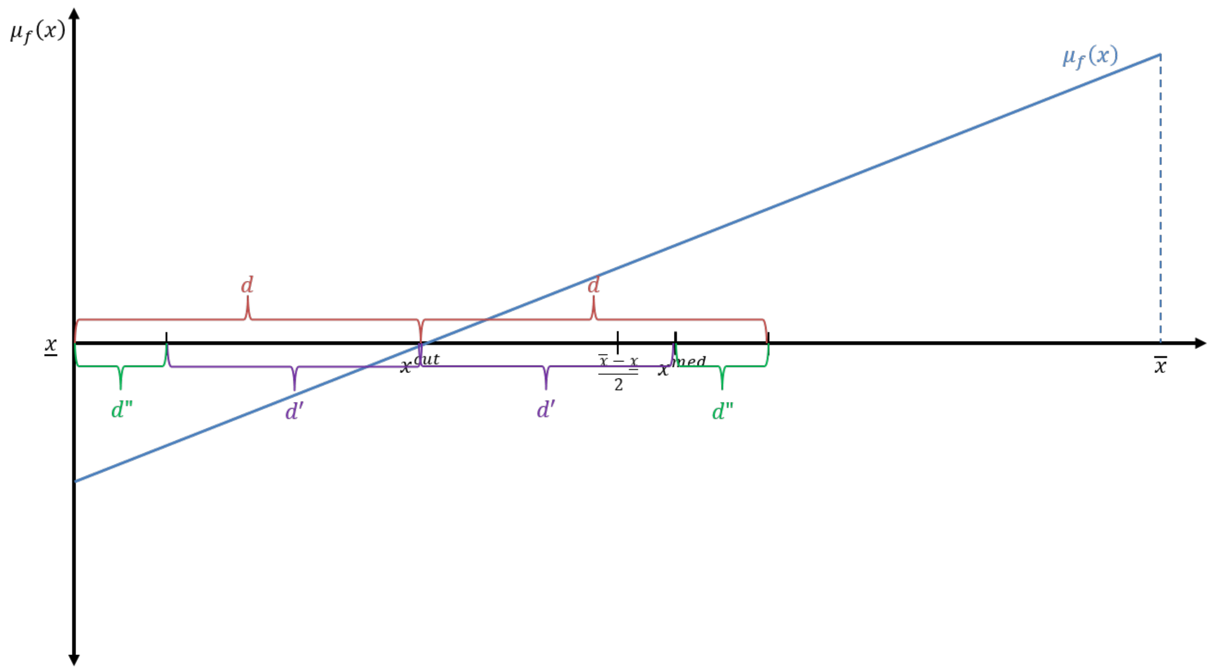

Figure 2: Linear Mean; $\nu_{f_n} \to \infty$

where the containment of x^i in $(x^{\text{cut}}, x^{\text{cut}} + d]$ follows from the definition of d. This means that as $n \to \infty$, however much turnout rises at any $x \in [\underline{x}, x^{\text{cut}})$, turnout falls by precisely the same amount at the corresponding point $[x^{\text{cut}} + |x^{\text{cut}} - x|] \in (x^{\text{cut}}, x^{\text{cut}} + d]$. Formally, the mass of agents i that turn out with $x \in [\underline{x}, x^{\text{cut}})$ in the first interval is

$$\int_{\underline{x}}^{x^{\text{cut}}} \text{Prob}_{f_n}[D^i - C^i > 0 \mid x^i = x]\, dx = \int_{\underline{x}}^{x^{\text{cut}}} (1 - \Phi_n^x(0))\, dx, \tag{7}$$

where Φ_n^x is the CDF of a normal variable with mean μ^x and variance ν_n. Meanwhile, the mass of agents i with $x^i \in (x^{\text{cut}}, x^{\text{cut}} + d]$ in the second interval that turn out is

$$\int_{x^{\text{cut}}}^{x^{\text{cut}}+d} \text{Prob}_{f_n}[D^i - C^i > 0 \mid x^i = x]\, dx = \int_{x^{\text{cut}}}^{x^{\text{cut}}+d} (1 - \Phi_n^x(0))\, dx. \tag{8}$$

Now, for $x \in (x^{\text{cut}}, x^{\text{cut}} + d]$, we have $\Phi_n^x(0) = 1 - \Phi_n^{x^{\text{cut}} - |x - x^{\text{cut}}|}(0)$ because $\mu^{x^{\text{cut}}} = 0$ and the normal distributions $\mathsf{N}(\mu^x, \nu_n)$ are symmetric about their means. Thus, as we increase n, the increase in turnout of agents i with $x^i \in [\underline{x}, x^{\text{cut}})$ exactly offsets the decrease in turnout of agents i with $x^i \in (x^{\text{cut}}, x^{\text{cut}} + d]$.

Meanwhile, the mass of agents i with $x^i \in (x^{\text{cut}} + d, \overline{x}]$ that turn out is

$$\int_{x^{\text{cut}}+d}^{\overline{x}} \text{Prob}_{f_n}[D^i - C^i > 0 \mid x^i = x]\, dx = \int_{x^{\text{cut}}+d}^{\overline{x}} (1 - \Phi_n^x(0))\, dx$$
$$= \int_{x^{\text{cut}}+d}^{\overline{x}} \left(\frac{1}{2} + \frac{1}{2}\text{erf}\left(\frac{-\mu^x}{\sqrt{2\nu_n}} \right) \right) dx, \qquad (9)$$

where erf is the error function for the normal distribution.[16] As $\mu^x > 0$ for $x \in (x^{\text{cut}} + d, \overline{x}]$, we see that (9) is decreasing in ν_n. (Note that this logic regarding the integrand in (9) implies that turnout monotonically converges to $\frac{1}{2}$ for all x.)

Combining the preceding observations, we see that, as $\nu_n \to \infty$ (or equivalently, as $n \to \infty$), the total mass of agents that turns out decreases monotonically to $\frac{d + \frac{1}{2}(\overline{x} - (x^{\text{cut}} + d))}{\overline{x} - \underline{x}}$,[17] with measure

$$\begin{aligned}
\frac{d + \frac{1}{2}(\overline{x} - (x^{\text{cut}} + d))}{\overline{x} - \underline{x}} &= \frac{1}{2}\left(\frac{2d + (\overline{x} - (x^{\text{cut}} + d))}{\overline{x} - \underline{x}} \right) \\
&= \frac{1}{2}\left(\frac{x^{\text{cut}} - \underline{x} + d + (\overline{x} - (x^{\text{cut}} + d))}{\overline{x} - \underline{x}} \right) \\
&= \frac{1}{2}\left(\frac{\overline{x} - \underline{x}}{\overline{x} - \underline{x}} \right) \\
&= \frac{1}{2}.
\end{aligned}$$

Now, let x_n^{med} be the preference of the median voter that turns out for a given n. Our assumption that $x^{\text{cut}} + d < \overline{x}$, combined with the preceding analysis, implies that $x_n^{\text{med}} > x^{\text{cut}}$ for all n. We let $d_n' = |x_n^{\text{med}} - x^{\text{cut}}|$. We suppose moreover that $d_n' < d$ for all n, and let $d_n'' = d - d_n'$. A variant of the preceding argument shows that as n increases:

1. The increase in turnout of agents i with $x^i \in [\underline{x} + d_n'', x^{\text{cut}}) = [x^{\text{cut}} - d_n', x^{\text{cut}})$ is exactly offset by the decrease in turnout of agents i with $x^i \in (x^{\text{cut}}, x^{\text{cut}} + d_n'] = (x^{\text{cut}}, x_n^{\text{med}}]$.

[16] As can be seen in Figure 2, for any $x \in (x^{\text{cut}} + d, \overline{x}]$, the integrand in (9) indicates how much of the distribution of $D^i - C^i$ falls above the axis; this, in turn, equals $\frac{1}{2}$ (for the part of the symmetric distribution above the μ^x line) plus the portion falling between that line and the axis. The error function for the stated value in (9) indicates how much of the density falls within that distance from the μ^x line (on either side of the μ^x line), so half of that value indicates the mass between the line and the axis.

[17] For the latter term in the numerator, because the erf in the integrand in (9) vanishes in the limit, we just have $\frac{1}{2}$ times the population fraction that falls in the interval given by the limits of integration. To understand the first term in the numerator, d, note that this is both the portion in the leftmost interval, $[\underline{x}, x^{\text{cut}})$, and those at the corresponding middle interval, $(x^{\text{cut}}, x^{\text{cut}} + d]$. Those at any x in the left interval turn out at a rate that is below $\frac{1}{2}$ by the same amount that those at the corresponding x in the middle interval turn out at a rate that is above $\frac{1}{2}$. Hence, for each corresponding pair of xs in the left and middle intervals, we have combined turnout of 1, and the interval has width of d.

2. The turnout of agents i with $x^i \in [\underline{x}, \underline{x} + d''_n)$—all of whom have $x^i < x_n^{\text{med}}$—increases.

3. The turnout of agents i with $x^i \in (x_n^{\text{med}}, \overline{x}]$—all of whom have $x^i > x_n^{\text{med}}$—decreases.

Observation 1 shows that changes in turnout of agents i with

$$x^i \in [\underline{x} + d''_n, x^{\text{cut}}) = [x^{\text{cut}} - d'_n, x^{\text{cut}})$$

and

$$x^i \in (x^{\text{cut}}, x^{\text{cut}} + d'_n] = (x^{\text{cut}}, x_n^{\text{med}}]$$

have no impact on the median voter, since all agents i with x^i in those intervals have $x^i < x_n^{\text{med}}$ and the total mass of such agents that turn out is constant. Observations 2 and 3, meanwhile, imply that the median voter x_n^{med} falls as n increases, since we see increased turnout of agents i with $x^i < x_n^{\text{med}}$ and decreased turnout of agents i with $x^i > x_n^{\text{med}}$. Combining the preceding three observations, we see that in this example, x_n^{med} falls monotonically and, in the limit as $n \to \infty$, x_n^{med} approaches the true median preference in the population. If $\mathcal{V}$ were a median voter mechanism, therefore, the voting outcome would monotonically approach the outcome that would arise with fully representative turnout. Note further that, in this example, turnout becomes increasingly representative in the broad sense that all groups with below-50% turnout (those to the left of x^{cut}) have monotonically increasing turnout and all those with above-50% turnout (to the right of x^{cut}) have monotonically decreasing turnout.[18] Finally, when f induces low turnout (below 50%), we have both turnout and representativeness rising with noise, but when f induces high turnout (above 50%), as in Figure 2, turnout falls with noise, yet representativeness still rises.

Further Examples

Finally, we consider briefly and informally a couple additional illustrations of the phenomena depicted in Proposition 3 and 4. Figure 3—which retains our uniform distribution of x^i from Figure 1 and Figure 2—depicts a situation in which the mean of $D^i - C^i$ conditional on x^i is higher at the extremes and, as drawn, to a greater extent near $x^i = \overline{x}$. In the Proposition 3 limit in which the variance goes to 0, only the extremes would turn out and, because of this skew, results under many standard voting mechanisms would be shifted strongly toward the right extreme outcome $\overline{x}$. Note that, as drawn, in the limit x_n^{med} would be to the right of x^{cut^2}. By contrast, in the Proposition 4 limit in which the variance goes to ∞, turnout would

[18] One may not, however, further infer that turnout as a whole becomes monotonically more representative because, for example, among those who turn out at a below-50% rate (to the left of x^{cut}), we have not shown that those who turn out less always have turnout rising more rapidly than those who turn out more.

be perfectly representative, with $x_n^{\text{med}} \to \frac{\overline{x}-\underline{x}}{2}$. Whether the limiting turnout level, also $\frac{1}{2}$, would be higher or lower than that implied by the figure would depend on further particulars. In general, the changes in both the turnout level and x_n^{med} need not monotonically approach their limiting values.

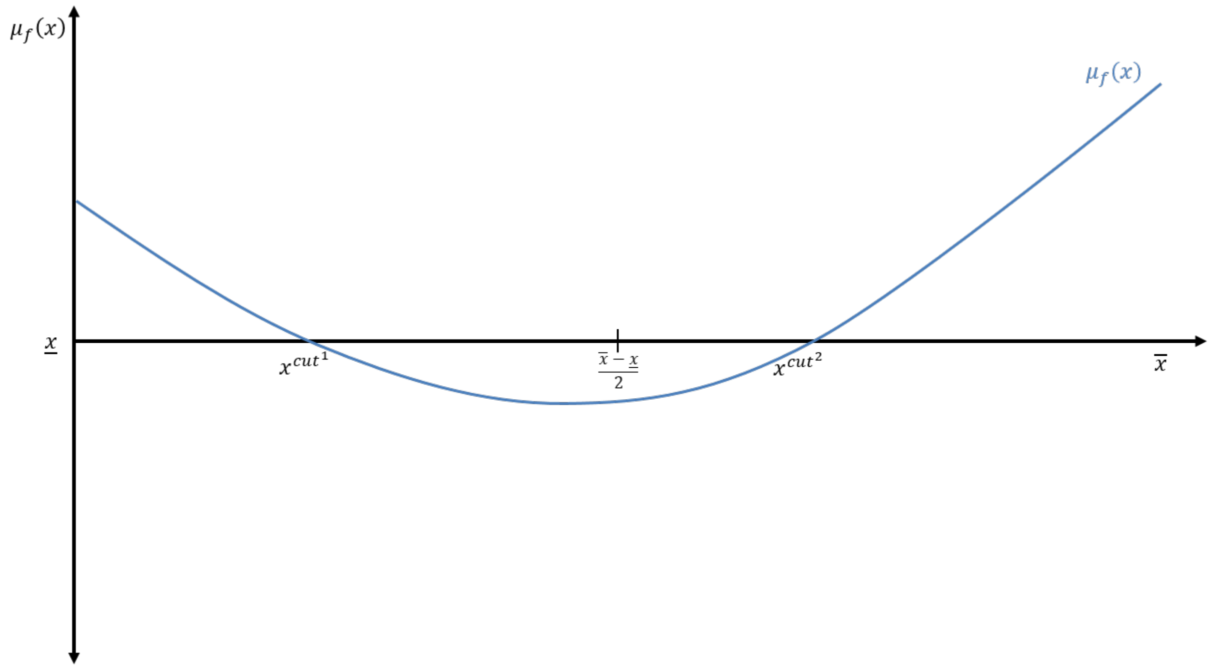

Figure 3: Nonlinear Mean; Extremists Turn Out

Now consider Figure 4, which is identical to Figure 3 except that everywhere the sign (but not the magnitude) of $\mu_f(x)$ is reversed. In the Proposition 3 limit, only the moderates now turn out and—opposite to what we had before—the skew is toward the left extreme $\underline{x}$. In this limit, however, we have x_n^{med} below $\frac{\overline{x}-\underline{x}}{2}$, but in the interval $(x^{\text{cut}^1}, x^{\text{cut}^2})$—and in that sense the skew is more moderate than in the prior case. Again, if we instead take the Proposition 4 limit, the turnout level approaches $\frac{1}{2}$ and the turnout set is fully representative, with $x_n^{\text{med}} \to \frac{\overline{x}-\underline{x}}{2}$.

Of course, Figures 1–4 depict just a few of the many possibilities. For example, if $\mu_f(x)$ were nearly horizontal and near the x-axis, perhaps a bit higher here and a bit lower there, then there might be little systematic skew in the turnout set regardless of the amount of noise. This case approximates that assumed in Proposition 1.

The key takeaway from all four propositions is that the conditional density function, $f(x^i, D^i, C^i)$, determines representativeness. It also determines the turnout level, but knowing the latter tells us little about the former unless we make further assumptions. Likewise, knowing how a reform influences the turnout level is not in general informative about how

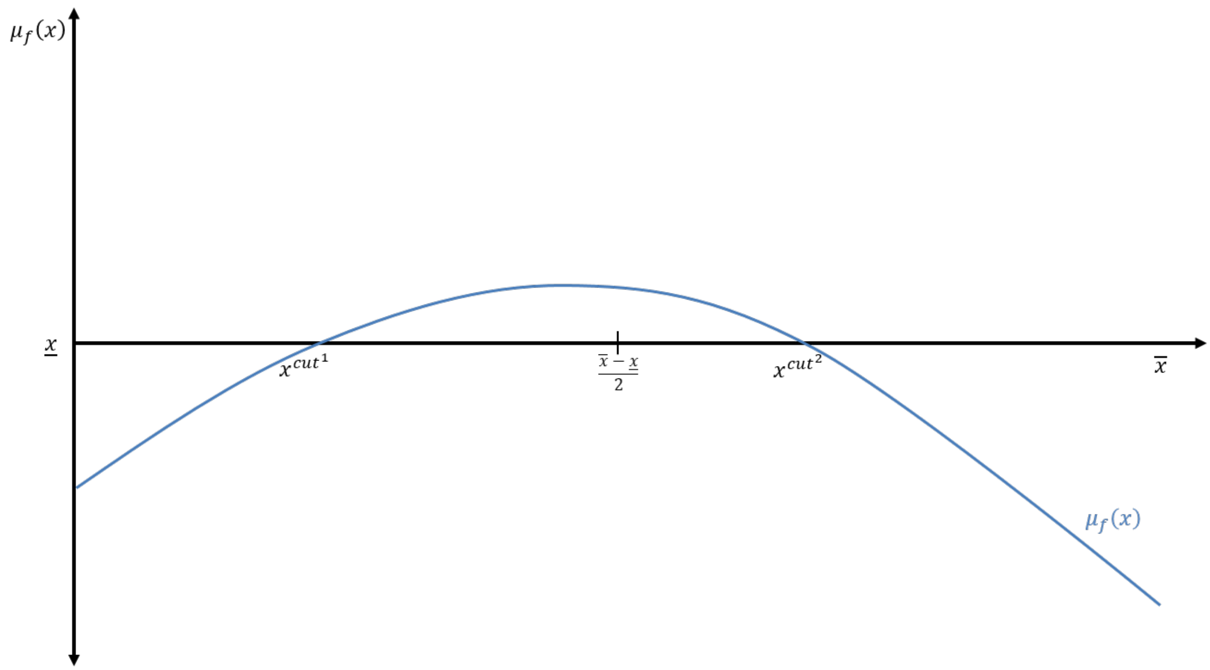

Figure 4: Nonlinear Mean; Moderates Turn Out

it influences representativeness.

4 Discussion

In our model, the probability of pivotality is literally 0, which the literature in section 2 suggests may often be a good approximation of reality when there is a reasonably large electorate. As a consequence, turnout is driven entirely by C and D. In this model, the properties of the joint density function, $f(x^i, D^i, C^i)$, determine the representativeness of the turnout set, $\widetilde{t_f}(x)$, which can differ radically from $\tilde{f}(x)$, the unconditional density of individuals' preferences. Our results cast prior theoretical and empirical research as well as policy analysis of voting reforms in a new light.

Regarding the theoretical literature, let us revisit Ledyard (1984), Myerson (2000), and Krishna and Morgan (2015). We previously noted that introducing D, enabling the generation of the substantial turnout levels observed in practice, eliminates the driving force in these models, which depends on turnout being very small and determined entirely by the relative magnitude of each individual's preference intensities and their Cs (for those whose C is in a neighborhood of 0). With D, and maintaining independence of x in the joint distribution, turnout is fully representative rather than intensity weighted. Of course, that outcome is only one of many possibilities. If independence is relaxed so that individuals' expressive utilities from voting or senses of duty, captured by D, are positively related to intensity, some degree of intensity-weighted turnout would be restored.[19] On the other hand, if individuals get greater expressive utility when casting a vote for candidates whose positions are closer to their own, as suggested by Brennan and Hamlin (1998), then those with the most extreme (and thus perhaps the most intense) preferences would be underrepresented in an election with two moderate choices; majority voting outcomes might then be further from the utilitarian (intensity-weighted) result than with full turnout, as assumed in simple voting models. Of course, D may instead be high among certain groups for all manner of reasons that have little to do with preference intensity and may or may not have a systematic relationship to preferences. For example, Ali and Lin (2013) suggest that individuals may vote to be seen as ethical; Harbaugh (1996), Gerber, Green, and Larimer (2008), Funk (2010), and DellaVigna, List, Malmendier, and Rao (2017) provide evidence that individuals are more likely to vote if others will learn that they did so; and Bond et al. (2012) show how individuals' social networks influence who votes. The nature of any dependence reflected in $f(x^i, D^i, C^i)$ presents an empirical question.

Much of the vast literature on the empirical determinants of voter turnout (see the Geys (2006) survey and Leighley and Nagler (2014) book) attempts to predict turnout *levels*. Our analysis suggests that this focus is misplaced if one is interested in representativeness and, ultimately, voting outcomes. Under the independence assumption underlying Proposition 1, the turnout level is irrelevant to representativeness and—under the scale-invariant voting mechanisms that are commonly employed in large elections—to outcomes. In the setting of our final proposition, increasing noise in the limit leads to full representativeness but turnout of only 50%—with the further implication that, if one began with higher, nonrepresentative turnout, that higher turnout level would be associated with less representativeness, not more.

Funk (2010) examines the move to optional voting by mail in Switzerland, a practice increasingly used in the United States. Funk finds that voting by mail not only reduced Cs but that it also must have reduced the social pressure on some individuals to vote (that is, their Ds), resulting in only a small and statistically insignificant net increase in the turnout level. We may also, however, be interested in how each component may have (differentially) influenced the composition of turnout. Reductions in C may have mattered most to indi-

[19]Likewise, if more informed voters, as in Feddersen and Pesendorfer (1996, 1999), Kim and Fey (2007), and McMurray (2013), also had a stronger sense of duty, they would be more likely to turn out. However, even if D were a precise reflection of preference intensity (or of informedness) and nothing else, the full utilitarian (informed) result would not arise because, with strictly positive D, the relevant range of C would no longer be confined to a neighborhood of 0, so the distribution of C in the relevant domain would no longer be (essentially) uniform. For example, if the distribution of C trails off at higher levels, intensity-based (or information-based) turnout selection would be attenuated as intensity (informedness) rises. That is, very low levels of intensity (or informedness) may be mostly screened out, with decreasing differentiation thereafter.

viduals with high time value (typically higher-income) or with less flexibility (those whose working environment and family responsibilities made it difficult to vote). The apparent reductions in D were shown to be the greatest in small and, one might suppose, close-knit communities, where being seen at the polls may previously have motivated turnout. Each of these groups may well have atypical preferences, in ways that do not offset each other.[20]

When turnout levels respond modestly to reforms or random shocks (such as the weather, as in Fowler (2015) and Hansford and Gomez (2010)), individuals whose voting behavior changes will be those near the margin, that is, with $D - C$ near 0. Without independence in the joint density function, these marginal voters will generally be an unrepresentative group. What matters more for representativeness, however, is how they compare to those who are inframarginal: who vote or do not vote, regardless. Consider our linear Example 1 and focus on Proposition 3's limiting case with no dispersion. In this special case, marginal voters (those at x^{cut}) will be toward the left extreme of the turnout set but toward the right extreme of the set of nonvoters, in both instances at the more moderate ends of those sets when considering the population as a whole. Under these assumptions, greater turnout of marginal voters would increase the representativeness of the turnout set.[21] Intuition from this sort of setting—combined with the simple point that, as turnout approaches 100%, the turnout set must eventually become representative—probably underlies the common supposition that higher turnout levels tend to be associated with more representative turnout. Our analysis identifies conditions under which turnout levels and representativeness tend to move together but also finds others in which they do not, including a simple setting in which they move in opposite directions (as illustrated in Example 2 after Proposition 4).

Consider now some important considerations outside of our model that are illuminated by our analysis. First, we assume throughout that different levels of C and D may be associated with different values of x but do not in any sense cause preferences to change; this juxtaposition is best appreciated by considering the case in which $\tilde{f}(x)$, the unconditional distribution of preferences, x, is fixed, but the joint density of x with C and D may change. When C or D move idiosyncratically and unexpectedly (say, due to the weather on election day), this interpretation may be convincing, and it further suggests that empirical work

[20] See also the work of Enos, Fowler, and Vavreck (2016), who show in a meta-analysis that get-out-the-vote experiments tend to increase the disparity in turnout between high- and low-propensity-to-vote individuals.

[21] Note that the change in representativeness would be misstated (and often overestimated) if one compared the typical traits of those who do turn out to the traits of those who do not, precisely because, as explained, most of these two groups are inframarginal and thus more extreme (in one direction or the other) than those whose participation is most likely to shift. For theoretical and empirical explorations in a similar spirit, see Grofman, Owen, and Collet (1999) and Brians and Grofman (1999), respectively. See also the work of Ansolabehere and Hersh (2012), who find that prior studies based on surveys overstate differences between voters and nonvoters.

using such instruments may plausibly identify the preferences of marginal voters, who in turn may be most influenced by some reforms. However, changing voting policy may directly influence the underlying distribution of x (the unconditional density). For example, many have suggested that, because voting is irrational when viewed purely in terms of individuals' private benefits from influencing the outcome, individuals who turn out must be motivated by D and hence might be expected to cast their votes based on social rather than narrower private preferences (see Goodin and Roberts (1975) and Margolis (1982)). However, if voting became extremely easy, individuals were paid to vote, or enforced compulsory voting were implemented, then $\tilde{f}(x)$, the unconditional distribution of preferences that votes actually reflect, may be directly influenced, in addition to D being altered, as noted above (see Ariely, Bracha, and Meier (2009), Frey and Jegen (2001), and Gneezy, Meier, and Rey-Biel (2011)). Voting policies—as well as electioneering (see Enos and Fowler (2018), Green and Gerber (2015) and Issenberg (2012))—that are aimed at any one of the three variables may influence the others or their interrelationships; moreover, these indirect effects could be more consequential than the direct effect of the intervention.

Our analysis also abstracts from what, precisely, is being voted on. If the choice is between candidates (or referenda that are costly to propose), our results have implications for positioning, an original focus of Ledyard (1984) that has been pursued in other literature. If candidates choose positions to maximize their chance of being elected, then in a two-candidate race under majority rule, they will choose the median location. Our work shows, however, that the relevant median is not the population median (which is what much applied work assumes) but rather the median of the turnout set, which our analysis aims to characterize by reference to the density function $f(x^i, D^i, C^i)$. One could also take into account that candidate positioning may influence various individuals' Ds, which would alter the turnout set and thereby the median voter in that set. This channel is a qualitatively different mechanism from that in prior work in which the influence of candidate positioning on turnout is entirely through the pivotality channel.

Our focus on how interdependencies reflected in $f(x^i, D^i, C^i)$ influence the representativeness of the turnout set also bears on structural features of voting systems. For example, the use of caucuses instead of primaries or the delegation of decisions (or influence) to committees often requires participants to incur a substantially higher C. A familiar benefit is that some of that cost involves becoming more informed, but our analysis highlights a different impact: if C is correlated with x, representativeness may be substantially altered. Indeed, the selection effect can be perverse: for example, if those willing to serve on a committee to design new laboratory facilities are those most interested in aesthetics relative to research, the committee might produce a building that is beautiful but unconducive to intellectual interchange.[22]

5 Conclusion

We analyze a model of voter turnout in which each individual's pivotality is literally 0 so that all of the action is in C and D. When preferences are uncorrelated with C and D, the level of turnout is irrelevant: as long turnout is positive, the turnout set is representative and the outcome is unaffected by the turnout level. If we instead admit arbitrary correlation between preferences, C, and D, then anything is possible: the distribution of preferences in the turnout set can be anything and the outcome can be anything, irrespective of the distribution of preferences in the population. Turnout sets and voting outcomes depend on the specific relationship between preferences, C, and D. Systematic skews can arise in straightforward ways (for example, if individuals with pro-left preferences tend to have lower Cs), but there is a tendency for noise in any such correlations to produce a more representative turnout set. To illustrate the potential disjunction between the level of turnout and turnout's representativeness, we provide a simple example in which, as noise increases, the level of turnout monotonically falls while representativeness monotonically rises.

We derive our results in a simple model so as to highlight our framing of the problem, which we regard as our main contribution. By analyzing the case in which an individual's probability of pivotality is 0 (approximating its negligible magnitude in many actual settings), we are led to focus on determinants of the turnout set that are qualitatively different from those that underlie much prior theoretical work. Moreover, in common with some of that research, our framework emphasizes the representativeness of the turnout set rather than the level of turnout, which can have different relationships to each other under different assumptions about the joint distribution $f(x^i, D^i, C^i)$.

As emphasized in our Introduction, we take no normative stance on the value of representativeness as such, particularly because of the known limitations of various voting rules with regard to preference intensity, voters' knowledge, and the manipulability of those preferences (along with the aforementioned potential endogeneity of expressed preferences to the voting mechanism itself). Indeed, when turnout is driven entirely by C and D, many of the potentially appealing features of pivotality-driven endogenous turnout—reflection of preference intensity and incorporation of information—vanish but others (high-D voters may be more inclined to vote in the social interest) may take their place. Regardless of whether representativeness is taken to be the primary objective or is regarded to be problematic in important ways, our approach leads to a different perspective on the analysis of voter turnout and

[22] See, for example, Osborne, Rosenthal, and Turner (2000) and Persico (2004).

emphasizes different empirical questions from those explored in much of the prior literature. Of particular interest is to understand how individuals' preferences (x) are related to their Cs and Ds as well as which individuals' Cs and Ds may be influenced by various policies.

6 References

Ali, S. Nageeb, and Charles Lin, Why People Vote: Ethical Motives and Social Incentives, *American Economic Journal: Microeconomics* 5: 73–98 (2013).

Ansolabehere, Stephen, and Eitan Hersh, Validation: What Big Data Reveal About Survey Misreporting and the Real Electorate, *Political Analysis* 20: 437–459 (2012).

Ariely, Dan, Anat Bracha, and Stephan Meier, Doing Good or Doing Well? Image Motivation and Monetary Incentives in Behaving Prosocially, *American Economic Review* 99: 544–555 (2009).

Bond, Robert M., et al., A 61-Million-Person Experiment in Social Influence and Political Mobilization, *Nature* 489: 295–298 (2012).

Brennan, Geoffrey, and Alan Hamlin, Expressive Voting and Electoral Equilibrium, *Public Choice* 95: 149–175 (1998).

Brians, Craig Leonard, and Bernard Grofman, When Registration Barriers Fall, Who Votes?: An Empirical Test of a Rational Choice Model, *Public Choice* 99: 161–176 (1999).

Chamberlain, Gary, and Michael Rothschild, A Note on the Probability of Casting a Decisive Vote, *Journal of Economic Theory* 25: 152–162 (1981).

Cox, Gary W., and Michael C. Munger, Closeness, Expenditures, and Turnout in the 1982 U.S. House Elections, *American Political Science Review* 83: 217–231 (1989).

DellaVigna, Stefano, John A. List, Ulrike Malmendier, and Gautam Rao, Voting to Tell Others, *Review of Economic Studies* 84: 143–181 (2017).

Downs, Anthony, *An Economic Theory of Democracy*, New York: Harper and Row (1957).

Enos, Ryan D., and Anthony Fowler, Pivotality and Turnout: Evidence from a Field Experiment in the Aftermath of a Tied Election, *Political Science Research and Methods* 2: 309–319 (2014).

__________ and __________, Aggregate Effects of Large-Scale Campaigns on Voter Turnout, *Political Science Research and Methods* 6: 773–751 (2018).

__________, __________, and Lynn Vavreck, Increasing Inequality: The Effect of GOTV Mobilization on the Composition of the Electorate, *Journal of Politics* 76: 273–288 (2014).

Feddersen, Timothy J., Rational Choice Theory and the Paradox of Not Voting, *Journal of Economic Perspectives* 18(1): 99–112 (2004).

__________ and Wolfgang Pesendorfer, The Swing Voter's Curse, *American Economic Review* 86: 408–424 (1996).

__________ and __________, Abstention in Elections with Asymmetric Information and Diverse Preferences, *American Political Science Review* 93: 381–398 (June 1999).

Fiorina, Morris P., Information and Rationality in Elections, in *Information and Democratic Processes*, John A. Ferejohn and James H. Kuklinski, eds., Urbana: University of Illinois Press, 329–342 (1990).

Fowler, Anthony, Regular Voters, Marginal Voters and the Electoral Effects of Turnout, *Political Science Research and Methods* 3: 205–219 (May 2015).

Frey, Bruno S., and Rito Jegen, Motivation Crowding Theory, *Journal of Economic Surveys* 15: 589–611 (2001).

Funk, Patricia, Social Incentives and Voter Turnout: Evidence from the Swiss Mail Ballot System, *Journal of the European Economic Association* 8: 1077–1103 (2010).

Gelman, Andrew, Nate Silver, and Aaron Edlin, What is the Probability Your Vote Will Make a Difference?, *Economic Inquiry* 50: 321–326 (2012).

Gerber, Alan S., Donald P. Green, and Christopher W. Larimer, Social Pressure and Voter Turnout: Evidence from a Large-Scale Field Experiment, *American Political Science Review* 102: 33–48 (2008).

__________, Mitchell Hoffman, John Morgan, and Collin Raymond, One in a Million: Field Experiments on Perceived Closeness of the Election and Voter Turnout, *NBER Working Paper* 23071 (2017).

Geys, Benny, Explaining Voter Turnout: A Review of Aggregate-level Research, *Electoral Studies* 25: 637–663 (2006).

Gneezy, Uri, Stephan Meier, and Pedro Rey-Biel, When and Why Incentives (Don't)

Work to Modify Behavior, *Journal of Economic Perspectives* 25(4): 191–210 (2011).

Goodin, R.E., & K.W.S. Roberts, The Ethical Voter, *American Political Science Review* 69: 926–928 (1975).

Green, Donald P., and Alan S. Gerber, *Get Out the Vote: How to Increase Voter Turnout*, 3rd ed., Washington, D.C.: Brookings Institution Press (2015)

Grofman, Bernard, Guillermo Owen, and Christian Collet, Rethinking the Partisan Effects of Higher Turnout: So What's the Question?, *Public Choice* 99: 357–376 (1999).

Hansford, Thomas G., and Brad T. Gomez, Estimating the Electoral Effects of Voter Turnout, *American Political Science Review* 104: 268–288 (2010).

Harbaugh, W.T., If People Vote Because They Like To, Then Why Do So Many of Them Lie?, *Public Choice* 89: 63–76 (1996).

Harsanyi, John C., Rational-Choice Models of Political Behavior vs. Functionalist and Conformist Theories, *World Politics* 21: 513–538 (1969).

Issenberg, Sasha, *The Victory Lab: The Secret Science of Winning Campaigns*, New York: Random House (2012).

Kim, Jaehoon, and Mark Fey, The Swing Voter's Curse with Adversarial Preferences, *Journal of Economic Theory* 135: 236–252 (2007).

Krishna, Vijay, and John Morgan, Majority Rule and Utilitarian Welfare, *American Economic Journal: Microeconomics* 7: 339–375 (2015).

Ledyard, John O., The Pure Theory of Large Two-candidate Elections, *Public Choice* 44: 7–41 (1984).

Leighley, Jan E., and Jonathan Nagler, *Who Votes Now? Demographics, Issues, Inequality, and Turnout in the United States*, Princeton: Princeton University Press (2014).

Margolis, Howard, *Selfishness, Altruism, and Rationality*, Chicago: University of Chicago Press (1982).

Martinelli, César, Would Rational Voters Acquire Costly Information?, *Journal of Economic Theory* 129: 225–251 (2006).

McMurray, Joseph C., Aggregating Information by Voting: The Wisdom of the Experts versus the Wisdom of the Masses, *Review of Economic Studies* 80: 277–312 (2013).

__________, The Paradox of Information and Voter Turnout, *Public Choice* 165: 13–23 (2015).

Mueller, Dennis, *Public Choice III*, Cambridge: Cambridge University Press (2003).

Mulligan, Casey B., and Charles G. Hunger, The Empirical Frequency of a Pivotal Vote, *Public Choice* 116: 31–54 (2003).

Myerson, Roger B., Large Poisson Games, *Journal of Economic Theory*: 94: 7–45 (2000).

Osborne, Martin J., Jeffrey S. Rosenthal, and Matthew A. Turner, Meetings with Costly Participation, *American Economic Review* 90: 927–43 (2000).

Palfrey, Thomas R., and Howard Rosenthal, Voter Participation and Strategic Uncertainty, *American Political Science Review* 79: 62–78 (1985).

Persico, Nicola, Committee Design with Endogenous Information, *Review of Economic Studies* 71: 165–191 (2004).

Riker, William H., and Peter C. Ordeshook, A Theory of the Calculus of Voting, *American Political Science Review* 62: 25–42 (1968).

Schwartz, Thomas, Your Vote Counts on Account of the Way it is Counted: An Institutional Solution to the Paradox of Voting, *Public Choice* 54: 101–121 (1987).